

# Uniform approximations by Fourier sums on classes of generalized Poisson integrals

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## Abstract

We find asymptotic equalities for exact upper bounds of approximations by Fourier sums in uniform metric on classes of  $2\pi$ -periodic functions, representable in the form of convolutions of functions  $\varphi$ , which belong to unit balls of spaces  $L_p$ , with generalized Poisson kernels. For obtained asymptotic equalities we introduce the estimates of remainder, which are expressed in the explicit form via the parameters of the problem.

**Key words:** Fourier sums, generalized Poisson integrals, asymptotic equality.

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**1. Introduction.** Let  $L_p$ ,  $1 \leq p < \infty$ , be the space of  $2\pi$ -periodic functions  $f$  summable to the power  $p$  on  $[0, 2\pi)$ , in which the norm is given by the formula  $\|f\|_p = \left( \int_0^{2\pi} |f(t)|^p dt \right)^{\frac{1}{p}}$ ;  $L_\infty$  be the space of measurable and essentially bounded  $2\pi$ -periodic functions  $f$  with the norm  $\|f\|_\infty = \operatorname{ess\,sup}_t |f(t)|$ ;  $C$  be the space of continuous  $2\pi$ -periodic functions  $f$ , in which the norm is specified by the equality  $\|f\|_C = \max_t |f(t)|$ .

Denote by  $C_{\beta,p}^{\alpha,r}$ ,  $\alpha > 0$ ,  $r > 0$ ,  $1 \leq p \leq \infty$ , the set of all  $2\pi$ -periodic functions, representable for all  $x \in \mathbb{R}$  as convolutions of the form (see, e.g., [1, p. 133])

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t) \varphi(t) dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B_p^0, \quad (1)$$

$$B_p^0 = \{ \varphi : \|\varphi\|_p \leq 1, \varphi \perp 1 \}, \quad 1 \leq p \leq \infty,$$

with fixed generated kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos \left( kt - \frac{\beta \pi}{2} \right), \quad \beta \in \mathbb{R}.$$

The kernels  $P_{\alpha,r,\beta}(t)$  are called generalized Poisson kernels. For  $r = 1$  and  $\beta = 0$  the kernels  $P_{\alpha,r,\beta}(t)$  are usual Poisson kernels of harmonic functions.

For any  $r > 0$  the classes  $C_{\beta,p}^{\alpha,r}$  belong to set of infinitely differentiable  $2\pi$ -periodic functions  $D^\infty$ , i.e.,  $C_{\beta,p}^{\alpha,r} \subset D^\infty$  (see, e.g., [1, p. 128], [2]). For  $r \geq 1$  the classes  $C_{\beta,p}^{\alpha,r}$  consist of functions  $f$ , admitting a regular extension into the strip  $|\operatorname{Im} z| \leq c$ ,  $c > 0$  in the complex plane (see, e.g., [1, p. 141]), i.e., are the classes of analytic functions. For

$r > 1$  the classes  $C_{\beta,p}^{\alpha,r}$  consist of functions regular on the whole complex plane, i.e., of entire functions (see, e.g., [1, p. 131]). Besides, it follows from the theorem 1 in [3] that for any  $r > 0$  the embedding holds  $C_{\beta,p}^{\alpha,r} \subset \mathcal{J}_{1/r}$ , where  $\mathcal{J}_a, a > 0$ , are known Gevrey classes

$$\mathcal{J}_a = \left\{ f \in D^\infty : \sup_{k \in \mathbb{N}} \left( \frac{\|f^{(k)}\|_C}{(k!)^a} \right)^{1/k} < \infty \right\}.$$

Approximation properties of classes of generalized Poisson integrals  $C_{\beta,p}^{\alpha,r}$  in metrics of spaces  $L_s, 1 \leq s \leq \infty$ , were considered in [4]–[12] from the viewpoint of order or asymptotic estimates for approximations by Fourier sums, best approximations and widths.

In the present paper we obtain asymptotic equalities as  $n \rightarrow \infty$  for the quantities

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = \sup_{f \in C_{\beta,p}^{\alpha,r}} \|f(\cdot) - S_{n-1}(f; \cdot)\|_C, \quad r > 0, \quad \alpha > 0, \quad 1 \leq p \leq \infty, \quad (2)$$

where  $S_{n-1}(f; \cdot)$  are the partial Fourier sums of order  $n - 1$  for a function  $f$ .

Approximation by Fourier sums on other classes of differentiable functions in uniform metric were investigated in works [1], [13]–[17].

Nikol'skii [14, p. 221] considered the case  $r = 1, p = \infty$  and established that following asymptotic equality is true

$$\mathcal{E}_n(C_{\beta,\infty}^{\alpha,1})_C = e^{-\alpha n} \left( \frac{8}{\pi^2} \mathbf{K}(e^{-\alpha}) + O(1)n^{-1} \right), \quad (3)$$

where

$$\mathbf{K}(q) := \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - q^2 \sin^2 t}}, \quad q \in (0, 1),$$

is a complete elliptic integral of the first kind, and  $O(1)$  is a quantity uniformly bounded in parameters  $n$  and  $\beta$ .

Later, the equality (3) was clarified by Stechkin [18, p. 139], who established the asymptotic formula

$$\mathcal{E}_n(C_{\beta,\infty}^{\alpha,1})_C = e^{-\alpha n} \left( \frac{8}{\pi^2} \mathbf{K}(e^{-\alpha}) + O(1) \frac{e^{-\alpha}}{(1 - e^{-\alpha})n} \right), \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad (4)$$

where  $O(1)$  is a quantity uniformly bounded in all analyzed parameters.

In work [10] for  $r = 1$  and arbitrary values of  $1 \leq p \leq \infty$  for quantities  $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C, \alpha > 0, \beta \in \mathbb{R}$ , the following equality was established

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,1})_C = e^{-\alpha n} \left( \frac{2}{\pi^{1+\frac{1}{p'}}} \|\cos t\|_{p'} K(p', e^{-\alpha}) + O(1) \frac{e^{-\alpha}}{n(1 - e^{-\alpha})^{s(p)}} \right), \quad (5)$$

where  $p' = \frac{p}{p-1}$ ,

$$s(p) := \begin{cases} 1, & p = \infty, \\ 2, & p \in [1, 2) \cup (2, \infty), \\ -\infty, & p = 2, \end{cases}$$

$$K(p', q) := \frac{1}{2^{1+\frac{1}{p'}}} \left\| (1 - 2q \cos t + q^2)^{-\frac{1}{2}} \right\|_{p'}, \quad q \in (0, 1),$$

and  $O(1)$  is a quantity uniformly bounded in  $n$ ,  $p$ ,  $\alpha$  and  $\beta$ . For  $p = \infty$ , by virtue of the known equality  $K(1, q) = \mathbf{K}(q)$ , the estimate (5) coincides with the estimate (4).

Note that for  $p = 2$  and  $r = 1$  formula (5) becomes the equality

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,1})_C = \frac{1}{\sqrt{\pi(1 - e^{-2\alpha})}} e^{-\alpha n}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad n \in \mathbb{N},$$

(see [10]). Moreover, it follows from [19] that for  $p = 2$  and  $r > 0$  for the quantities  $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$  the equalities take place

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C = \frac{1}{\sqrt{\pi}} \left( \sum_{k=n}^{\infty} e^{-2\alpha k^r} \right)^{\frac{1}{2}}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad n \in \mathbb{N}. \quad (6)$$

In the case of  $r > 1$  and  $p = \infty$  the asymptotic equalities for the quantities  $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , were obtained by Stepanets [20, Chapter 3, Section 9], who showed that for any  $n \in \mathbb{N}$

$$\mathcal{E}_n(C_{\beta,\infty}^{\alpha,r})_C = \left( \frac{4}{\pi} + \gamma_n \right) e^{-\alpha n^r}, \quad (7)$$

where

$$|\gamma_n| < 2 \left( 1 + \frac{1}{\alpha r n^{r-1}} \right) e^{-\alpha r n^{r-1}}.$$

Later Telyakovskii [6] established the asymptotic equality

$$\mathcal{E}_n(C_{\beta,\infty}^{\alpha,r})_C = \frac{4}{\pi} e^{-\alpha n^r} + O(1) \left( e^{-\alpha(2(n+1)^r - n^r)} + \left( 1 + \frac{1}{\alpha r(n+2)^r} \right) e^{-\alpha(n+2)^r} \right), \quad (8)$$

where  $O(1)$  is a quantity uniformly bounded in all analyzed parameters. Formula (8) contains more exact estimate of remainder in asymptotic decomposition of the quantity  $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$  comparing with the estimate (7).

For  $r > 1$  and for arbitrary values of  $1 \leq p \leq \infty$  the asymptotic equalities for the quantities  $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , are found in [10] and have the form

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = e^{-\alpha n^r} \left( \frac{\|\cos t\|_{p'}}{\pi} + O(1) \left( 1 + \frac{1}{\alpha r n^{r-1}} \right) e^{-\alpha n^{r-1}} \right), \quad (9)$$

where  $O(1)$  is a quantity uniformly bounded in all analyzed parameters. For  $p = \infty$  the formula (9) follows from (7) and (8).

Concerning the case  $0 < r < 1$ , except the presented above case  $p = 2$ , asymptotic equalities for quantities  $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , were known only for  $p = \infty$  due to the work of Stepanets [21], who showed that

$$\mathcal{E}_n(C_{\beta,\infty}^{\alpha,r})_C = \frac{4}{\pi^2} e^{-\alpha n^r} \ln n^{1-r} + O(1) e^{-\alpha n^r}, \quad (10)$$

where  $O(1)$  is a quantity uniformly bounded in  $n$  and  $\beta$ .

In case of  $0 < r < 1$  and  $1 \leq p < \infty$  the following order estimates for quantities  $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , hold (see, e.g., [8], [11])

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C \asymp e^{-\alpha n^r} n^{\frac{1-r}{p}}. \quad (11)$$

We remark that for  $0 < r < 1$  and  $1 \leq p < \infty$  Fourier sums provide the order of best approximations of classes  $C_{\beta,p}^{\alpha,r}$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , in uniform metric, i.e. (see, e.g., [[11], [12]])

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C \asymp E_n(C_{\beta,p}^{\alpha,r})_C \asymp e^{-\alpha n^r} n^{\frac{1-r}{p}},$$

where

$$E_n(C_{\beta,p}^{\alpha,r})_C = \sup_{f \in C_{\beta,p}^{\alpha,r}} \inf_{t_{n-1} \in \mathcal{T}_{2n-1}} \|f - t_{n-1}\|_C,$$

and  $\mathcal{T}_{2n-1}$  is the subspace of all trigonometric polynomials  $t_{n-1}$  of degree not higher than  $n-1$ .

Besides, as follows from Temlyakov's work [8] for  $2 \leq p < \infty$  quantities of approximations by Fourier sums realize order of the linear widths  $\lambda_{2n}$  (definition of  $\lambda_m$  see, e.g., [22, Chapter 1, Section 1.2]) of the classes  $C_{0,p}^{\alpha,r}$ , i.e.

$$\lambda_{2n}(C_{0,p}^{\alpha,r}, C) \asymp \mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C.$$

In this paper we establish asymptotically sharp estimates of the quantities  $\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ , for any  $0 < r < 1$  i  $1 \leq p \leq \infty$ . In particular, it is proved, that for  $r \in (0, 1)$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$  and  $1 < p \leq \infty$  as  $n \rightarrow \infty$  the following asymptotic equality takes place

$$\begin{aligned} \mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C &= e^{-\alpha n^r} n^{\frac{1-r}{p}} \left( \frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} \left( \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} + \right. \\ &\quad \left. O(1) \left( \frac{1}{n^{(1-r)(p'-1)}} + \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right) \right), \end{aligned} \quad (12)$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ , and  $O(1)$  is a quantity uniformly bounded with respect to  $n$  and  $\beta$ . Herewith, in this paper we found the estimates for remainder in (12), which are expressed via the parameters of the problem  $\alpha, r, p$  in the explicit form and that can be used for practical application.

**2. Formulation of main results.** For arbitrary  $v > 0$  and  $1 \leq s \leq \infty$  assume

$$J_s(v) := \left\| \frac{1}{\sqrt{t^2 + 1}} \right\|_{L_s[0,v]}, \quad (13)$$

where

$$\|f\|_{L_s[a,b]} = \begin{cases} \left( \int_a^b |f(t)|^s dt \right)^{\frac{1}{s}}, & 1 \leq s < \infty, \\ \operatorname{ess\,sup}_{t \in [a,b]} |f(t)|, & s = \infty. \end{cases}$$

Also for  $\alpha > 0$ ,  $r \in (0, 1)$  and  $1 \leq p \leq \infty$  we denote by  $n_0 = n_0(\alpha, r, p)$  the smallest integer  $n$  such that

$$\frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r \chi(p)}{n^{1-r}} \leq \begin{cases} \frac{1}{14}, & p = 1, \\ \frac{1}{(3\pi)^3} \cdot \frac{p-1}{p}, & 1 < p < \infty, \\ \frac{1}{(3\pi)^3}, & p = \infty, \end{cases} \quad (14)$$

where  $\chi(p) = p$  for  $1 \leq p < \infty$  and  $\chi(p) = 1$  for  $p = \infty$ .

With the notations introduced above, the main result of this paper is formulated in the following statement:

**Theorem 1.** *Let  $0 < r < 1$ ,  $1 \leq p \leq \infty$ ,  $\alpha > 0$  and  $\beta \in \mathbb{R}$ . Then for  $n \geq n_0(\alpha, r, p)$  the following estimate is true*

$$\begin{aligned} \mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C &= e^{-\alpha n^r} n^{\frac{1-r}{p}} \left( \frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \right. \\ &\quad \left. + \gamma_{n,p}^{(1)} \left( \frac{1}{(\alpha r)^{1+\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right) \right), \end{aligned} \quad (15)$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ , and the quantity  $\gamma_{n,p}^{(1)} = \gamma_{n,p}^{(1)}(\alpha, r, \beta)$  is such that  $|\gamma_{n,p}^{(1)}| \leq (14\pi)^2$ .

Now we present some corollaries of theorem 1.

For  $1 < p < \infty$  theorem 1 yields the following statement:

**Theorem 2.** *Let  $0 < r < 1$ ,  $1 \leq p < \infty$ ,  $\alpha > 0$  and  $\beta \in \mathbb{R}$ . Then for  $1 < p < \infty$  and  $n \geq n_0(\alpha, r, p)$  the following estimate is true*

$$\begin{aligned} \mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C &= e^{-\alpha n^r} n^{\frac{1-r}{p}} \left( \frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} \left( \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} + \right. \\ &\quad \left. + \gamma_{n,p}^{(2)} \left( \frac{1}{p' - 1} \frac{(\alpha r)^{\frac{p'-1}{p}}}{n^{(1-r)(p'-1)}} + \frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}} \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right) \right), \end{aligned} \quad (16)$$

and for  $p = 1$  and  $n \geq n_0(\alpha, r, 1)$  the estimate is true

$$\mathcal{E}_n(C_{\beta,1}^{\alpha,r})_C = e^{-\alpha n^r} n^{1-r} \left( \frac{1}{\pi \alpha r} + \gamma_{n,1}^{(2)} \left( \frac{1}{(\alpha r)^2} \frac{1}{n^r} + \frac{1}{n^{1-r}} \right) \right), \quad (17)$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ , and the quantity  $\gamma_{n,p}^{(2)} = \gamma_{n,p}^{(2)}(\alpha, r, \beta)$  is such that  $|\gamma_{n,p}^{(2)}| \leq (14\pi)^2$ .

**Proof of the theorem 2.** According to theorem 1 the following estimate is true for all  $1 < p < \infty$ ,  $0 < r < 1$ ,  $\alpha > 0$ ,  $\beta \in \mathbb{R}$  and  $n \geq n_0(\alpha, r, p)$

$$\begin{aligned} \mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C &= e^{-\alpha n^r} n^{\frac{1-r}{p}} \left( \frac{\|\cos t\|_{p'}}{(\alpha r)^{\frac{1}{p}} \pi^{1+\frac{1}{p'}}} \left( \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} + \right. \\ &\quad \left. + \gamma_{n,p}^{(1)} \left( \frac{1}{(\alpha r)^{1+\frac{1}{p}}} \left( \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right) \right), \end{aligned} \quad (18)$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ , and the quantity  $\gamma_{n,p}^{(1)} = \gamma_{n,p}^{(1)}(\alpha, r, \beta)$  is such that  $|\gamma_{n,p}^{(1)}| \leq (14\pi)^2$ .

By applying the Lagrange theorem, for  $n \geq n_0(\alpha, r, p)$  we obtain

$$\begin{aligned}
& \left( \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} - \left( \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} \leq \\
& \leq \frac{1}{p'} \left( \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'} - 1} \int_{\frac{\pi n^{1-r}}{\alpha r}}^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \leq \frac{1}{p'} \left( \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t + 1)^{p'}} \right)^{-\frac{1}{p}} \int_{\frac{\pi n^{1-r}}{\alpha r}}^\infty \frac{dt}{t^{p'}} = \\
& = \frac{1}{p'} \frac{1}{(p' - 1)} \left( 1 - \left( \frac{\pi n^{1-r}}{\alpha r} + 1 \right)^{1-p'} \right)^{-\frac{1}{p}} \frac{1}{(p' - 1)^{\frac{1}{p}}} \left( \frac{\alpha r}{\pi n^{1-r}} \right)^{p'-1} \leq \\
& \leq \frac{1}{p'} \frac{1}{(p' - 1)} \left( 1 - \left( 27\pi^4 \frac{p^2}{p-1} + 1 \right)^{1-p'} \right)^{-\frac{1}{p}} \frac{1}{(p' - 1)^{\frac{1}{p}}} \left( \frac{\alpha r}{\pi n^{1-r}} \right)^{p'-1} = \\
& = \frac{1}{p' - 1} \left( \frac{\alpha r}{\pi n^{1-r}} \right)^{p'-1} \frac{(p-1)^{\frac{p-1}{p}}}{p} \left( 1 - \left( 27\pi^4 \frac{p^2}{p-1} + 1 \right)^{\frac{1}{1-p}} \right)^{-\frac{1}{p}}. \tag{19}
\end{aligned}$$

It can be shown that

$$\frac{(p-1)^{\frac{p-1}{p}}}{p} \left( 1 - \left( 27\pi^4 \frac{p^2}{p-1} + 1 \right)^{\frac{1}{1-p}} \right)^{-\frac{1}{p}} < 2.$$

As follows from (19)

$$\begin{aligned}
& \left( \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} = \\
& = \left( \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} + \frac{\Theta_{\alpha,r,p,n}^{(1)}}{p' - 1} \left( \frac{\alpha r}{\pi n^{1-r}} \right)^{p'-1}, \quad |\Theta_{\alpha,r,p,n}^{(1)}| < 2. \tag{20}
\end{aligned}$$

From relations

$$\left( \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} \leq \left( \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} < \left( 1 + \int_1^\infty \frac{dt}{t^{p'}} \right)^{\frac{1}{p'}} < p^{\frac{1}{p'}} \tag{21}$$

and formulas (18) and (20) we obtain (16).

Formula (17) can be obtained from the equality (15) as consequence of substitution  $p = 1$  and elementary transformations. Theorem 2 is proved.

The following statement follows from the theorem 2 in the case  $p = 2$ .

**Corollary 1.** *Let  $0 < r < 1$ ,  $\alpha > 0$  and  $\beta \in \mathbb{R}$ . Then for  $n \geq n_0(\alpha, r, 2)$  the following estimate is true*

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C = \frac{e^{-\alpha n^r}}{\sqrt{2\pi\alpha r}} n^{\frac{1-r}{2}} \left( 1 + \gamma_n^{(1)} \left( \frac{1}{\alpha r} \frac{1}{n^r} + \frac{\sqrt{\alpha r}}{n^{\frac{1-r}{2}}} \right) \right), \tag{22}$$

where the quantity  $\gamma_n^{(1)} = \gamma_n^{(1)}(\alpha, r, \beta)$  is such that  $|\gamma_n^{(1)}| \leq 392\pi^{\frac{5}{2}}$ .

**Proof of the corollary 1.** Indeed, setting  $p = p' = 2$  in the equality (16), we obtain for  $n \geq n_0(\alpha, r, 2)$

$$\begin{aligned} \mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C &= e^{-\alpha n^r} n^{\frac{1-r}{2}} \left( \frac{\|\cos t\|_2}{\pi^{\frac{3}{2}}(\alpha r)^{\frac{1}{2}}} \left( \int_0^\infty \frac{dt}{t^2 + 1} \right)^{\frac{1}{2}} + \right. \\ &\quad \left. + \gamma_{n,2}^{(2)} \left( \frac{\sqrt{\alpha r}}{n^{1-r}} + \frac{\sqrt{2}}{(\alpha r)^{\frac{3}{2}}} \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{2}}} \right) \right) = \\ &= \frac{e^{-\alpha n^r}}{\sqrt{2\pi\alpha r}} n^{\frac{1-r}{2}} \left( 1 + \gamma_{n,2}^{(2)} \sqrt{2\pi} \left( \frac{\alpha r}{n^{1-r}} + \frac{\sqrt{2}}{\alpha r} \frac{1}{n^r} + \frac{\sqrt{\alpha r}}{n^{\frac{1-r}{2}}} \right) \right). \end{aligned} \quad (23)$$

According to (14) for  $n \geq n_0(\alpha, r, 2)$

$$\frac{\sqrt{\alpha r}}{n^{\frac{1-r}{2}}} \leq \frac{1}{2(3\pi)^{\frac{3}{2}}},$$

therefore

$$\left( \frac{\alpha r}{n^{1-r}} + \frac{\sqrt{2}}{\alpha r} \frac{1}{n^r} + \frac{\sqrt{\alpha r}}{n^{\frac{1-r}{2}}} \right) \leq \sqrt{2} \left( \frac{1}{\alpha r} \frac{1}{n^r} + \frac{\sqrt{\alpha r}}{n^{\frac{1-r}{2}}} \right). \quad (24)$$

From (23) and (24) we have (22). Corollary 1 is proved.

However, it is possible to obtain more exact estimate than (22) on the basis of equality (6). Namely, for  $\alpha > 0$ ,  $r \in (0, 1)$ ,  $\beta \in \mathbb{R}$  and  $n \geq n_0(\alpha, r, 2)$  the following estimate is true

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C = \frac{e^{-\alpha n^r}}{\sqrt{2\pi\alpha r}} n^{\frac{1-r}{2}} \left( 1 + \gamma_n^{(2)} \left( \frac{1}{2\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \quad (25)$$

where the quantity  $\gamma_n^{(2)} = \gamma_n^{(2)}(\alpha, r)$  is such that  $|\gamma_n^{(2)}| \leq \sqrt{\frac{54\pi^3}{54\pi^3 - 1}}$ . In order to prove (25) we use the following estimate, which will be useful in what follows.

Let  $\gamma > 0$ ,  $r > 0$ ,  $m \geq 1$  and  $\delta \in \mathbb{R}$ . Then for  $m \geq \left( \frac{14|\delta+1-r|}{\gamma r} \right)^{\frac{1}{r}}$  the estimate takes place

$$\int_m^\infty e^{-\gamma t^r} t^\delta dt = \frac{e^{-\gamma m^r}}{\gamma r} m^{\delta+1-r} \left( 1 + \Theta_{\gamma,m}^{r,\delta} \frac{|\delta+1-r|}{\gamma r} \frac{1}{m^r} \right), \quad |\Theta_{\gamma,m}^{r,\delta}| \leq \frac{14}{13}. \quad (26)$$

Indeed, integrating by parts, we obtain

$$\int_m^\infty e^{-\gamma t^r} t^\delta dt = \frac{e^{-\gamma m^r}}{\gamma r} m^{\delta+1-r} + \frac{\delta+1-r}{\gamma r} \int_m^\infty e^{-\gamma t^r} t^{-r+\delta} dt. \quad (27)$$

Since

$$\int_m^\infty e^{-\gamma t^r} t^{-r+\delta} dt = \frac{\overline{\Theta}_{\gamma,m}^{r,\delta}}{m^r} \int_m^\infty e^{-\gamma t^r} t^\delta dt, \quad 0 < \overline{\Theta}_{\gamma,m}^{r,\delta} < 1, \quad (28)$$

by virtue of (27) for  $m \geq \left(\frac{14|\delta+1-r|}{\gamma r}\right)^{\frac{1}{r}}$  we have

$$\int_m^\infty e^{-\gamma t^r} t^\delta dt \leq \frac{e^{-\gamma m^r}}{\gamma r} m^{\delta+1-r} + \frac{1}{14} \int_m^\infty e^{-\gamma t^r} t^\delta dt,$$

whence

$$\int_m^\infty e^{-\gamma t^r} t^\delta dt \leq \frac{14e^{-\gamma m^r}}{13\gamma r} m^{\delta+1-r}. \quad (29)$$

The estimate (26) follows from (27)–(29).

From the equality (6) and relation

$$\int_n^\infty \xi(u) du < \sum_{j=n}^\infty \xi(j) < \int_n^\infty \xi(u) du + \xi(n), \quad (30)$$

which takes place for any positive and decreasing function  $\xi(u)$ ,  $u \geq 1$ , such that  $\int_n^\infty \xi(u) du < \infty$ , we get

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C = \frac{1}{\sqrt{\pi}} \left( \int_n^\infty e^{-2\alpha t^r} dt + \Theta_{\alpha,r,n}^{(1)} e^{-2\alpha n^r} \right)^{\frac{1}{2}}, \quad |\Theta_{\alpha,r,n}^{(1)}| < 1. \quad (31)$$

In order to estimate the integral  $\int_n^\infty e^{-2\alpha t^r} dt$  it suffices to use the equality (26) for  $\gamma = 2\alpha$ ,

$\delta = 0$ ,  $m = n$  and  $r \in (0, 1)$ . Then, taking into account that  $n_0(\alpha, r, 2) > \left(\frac{7(1-r)}{\alpha r}\right)^{\frac{1}{r}}$ , for  $n \geq n_0(\alpha, r, 2)$  from (26) and (31) we get

$$\begin{aligned} \mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C &= \frac{1}{\sqrt{\pi}} \left( \frac{e^{-2\alpha n^r}}{2\alpha r} n^{1-r} \left( 1 + \Theta_{\alpha,r,n}^{r,0} \frac{(1-r)}{2\alpha r} \frac{1}{n^r} \right) + \Theta_{\alpha,r,n}^{(1)} e^{-2\alpha n^r} \right)^{\frac{1}{2}} = \\ &= \frac{e^{-\alpha n^r}}{\sqrt{2\pi\alpha r}} n^{\frac{1-r}{2}} \left( 1 + \Theta_{\alpha,r,n}^{(2)} \left( \frac{1}{2\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right)^{\frac{1}{2}}, \quad |\Theta_{\alpha,r,n}^{(2)}| \leq 2. \end{aligned} \quad (32)$$

Since for  $n > n_0(\alpha, r, 2)$

$$\begin{aligned} &\left| \left( 1 + \Theta_{\alpha,r,n}^{(2)} \left( \frac{1}{2\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right)^{\frac{1}{2}} - 1^{\frac{1}{2}} \right| \leq \\ &\leq \frac{1}{\sqrt{1 - \left( \frac{1}{\alpha r} \frac{1}{n^r} + \frac{2\alpha r}{n^{1-r}} \right)}} \left( \frac{1}{2\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \leq \sqrt{\frac{54\pi^3}{54\pi^3 - 1}} \left( \frac{1}{2\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right), \end{aligned}$$

then (25) follows from (32).

In the case of  $p = \infty$  theorem 1 allows to clarify the asymptotic equality (10).



We set  $n_1 = n_1(\alpha, r)$  be the smallest number  $n$  such that

$$\frac{1}{\alpha r} \frac{1}{n^r} \left( 1 + \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) \right) + \frac{\alpha r}{n^{1-r}} \leq \frac{1}{(3\pi)^3}. \quad (33)$$

The following assertion takes place.

**Theorem 3.** *Let  $0 < r < 1$ ,  $\alpha > 0$  and  $\beta \in \mathbb{R}$ . Then for  $n \geq n_1(\alpha, r)$  the following estimate is true*

$$\mathcal{E}_n(C_{\beta, \infty}^{\alpha, r})_C = \frac{4}{\pi^2} e^{-\alpha n^r} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \gamma_{n, \infty}^{(2)} e^{-\alpha n^r}, \quad (34)$$

where the quantity  $\gamma_{n, \infty}^{(2)} = \gamma_{n, \infty}^{(2)}(\alpha, r, \beta)$  is such that  $|\gamma_{n, \infty}^{(2)}| \leq 20\pi^4$ .

**Proof of the theorem 3.** From definitions (33) and (14) it follows that  $n_1(\alpha, r) > n_0(\alpha, r, \infty)$ . So, applying the equality (15) for  $p = \infty$  ( $p' = 1$ ), we get for  $n \geq n_1(\alpha, r)$

$$\mathcal{E}_n(C_{\beta, \infty}^{\alpha, r})_C = e^{-\alpha n^r} \left( \frac{4}{\pi^2} \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2 + 1}} + \gamma_{n, \infty}^{(1)} \left( \frac{1}{\alpha r} \frac{1}{n^r} \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2 + 1}} + 1 \right) \right). \quad (35)$$

Since

$$\begin{aligned} \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2 + 1}} &= \int_1^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{t} + \left( \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2 + 1}} - \int_1^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{t} \right) = \\ &= \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \Theta_{\alpha, r, n}^{(3)}, \quad 0 < \Theta_{\alpha, r, n}^{(3)} < 1, \end{aligned} \quad (36)$$

by virtue of (35) and (36) for  $n \geq n_1(\alpha, r)$

$$\begin{aligned} \mathcal{E}_n(C_{\beta, \infty}^{\alpha, r})_C &= e^{-\alpha n^r} \left( \frac{4}{\pi^2} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \frac{4}{\pi^2} \Theta_{\alpha, r, n}^{(3)} + \right. \\ &\quad \left. + \gamma_{n, \infty}^{(1)} \left( \frac{1}{\alpha r} \frac{1}{n^r} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \frac{\Theta_{\alpha, r, n}^{(3)}}{\alpha r n^r} + 1 \right) \right). \end{aligned} \quad (37)$$

The results of our calculations show that for  $n \geq n_1(\alpha, r)$

$$\frac{4}{\pi^2} \Theta_{\alpha, r, n}^{(3)} + |\gamma_{n, \infty}^{(1)}| \left( \frac{1}{\alpha r} \frac{1}{n^r} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \frac{\Theta_{\alpha, r, n}^{(3)}}{\alpha r n^r} + 1 \right) \leq 20\pi^4, \quad (38)$$

and therefore, in view of (37) and (38) we obtain (34). Theorem 3 is proved.

The asymptotic equality (10), which was established by Stepanets, follows from the relation (34).

**3. Proof of the theorem 1.** According to (1) and (2) we have

$$\mathcal{E}_n(C_{\beta, p}^{\alpha, r})_C = \frac{1}{\pi} \sup_{\varphi \in B_p^0} \left\| \int_{-\pi}^{\pi} P_{\alpha, r, \beta}^{(n)}(x - t) \varphi(t) dt \right\|_C, \quad 1 \leq p \leq \infty, \quad (39)$$

where

$$P_{\alpha,r,\beta}^{(n)}(t) := \sum_{k=n}^{\infty} e^{-\alpha k^r} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad 0 < r < 1, \quad \alpha > 0, \quad \beta \in \mathbb{R}. \quad (40)$$

Taking into account the invariance of the sets  $B_p^0$ ,  $1 \leq p \leq \infty$ , under shifts of the argument, from (39) we conclude that

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = \frac{1}{\pi} \sup_{\varphi \in B_p^0} \int_{-\pi}^{\pi} P_{\alpha,r,\beta}^{(n)}(t) \varphi(t) dt. \quad (41)$$

On the basis of the duality relation (see, e.g., [22, Chapter 1, Section 1.4])

$$\sup_{\varphi \in B_p^0} \int_{-\pi}^{\pi} P_{\alpha,r,\beta}^{(n)}(t) \varphi(t) dt = \inf_{\lambda \in \mathbb{R}} \|P_{\alpha,r,\beta}^{(n)}(t) - \lambda\|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (42)$$

In order to find the estimate for the quantity  $\inf_{\lambda \in \mathbb{R}} \|P_{\alpha,r,\beta}^{(n)}(t) - \lambda\|_{p'}$  we use the following assertion, proof of which will be presented later.

**Lemma 1.** *Let  $1 \leq s \leq \infty$ ,  $2\pi$ -periodic functions  $g(t)$  and  $h(t)$  have finite derivatives and satisfy the conditions:*

$$r(t) := \sqrt{g^2(t) + h^2(t)} \neq 0, \quad (43)$$

$$M := \sup_{t \in \mathbb{R}} \frac{\sqrt{(g'(t))^2 + (h'(t))^2}}{\sqrt{g^2(t) + h^2(t)}} < \infty. \quad (44)$$

Then for the function

$$\phi(t) = g(t) \cos(nt + \gamma) + h(t) \sin(nt + \gamma), \quad \gamma \in \mathbb{R}, \quad n \in \mathbb{N}, \quad (45)$$

for all numbers  $n \geq \begin{cases} 4\pi s M, & 1 \leq s < \infty, \\ 1, & s = \infty, \end{cases}$  the following estimates take place

$$\|\phi\|_s = \|r\|_s \left( \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} + \delta_{s,n}^{(1)} \frac{M}{n} \right), \quad (46)$$

$$\inf_{\lambda \in \mathbb{R}} \|\phi(t) - \lambda\|_s = \|r\|_s \left( \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} + \delta_{s,n}^{(2)} \frac{M}{n} \right), \quad (47)$$

$$\sup_{h \in \mathbb{R}} \frac{1}{2} \|\phi(t+h) - \phi(t)\|_s = \|r\|_s \left( \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} + \delta_{s,n}^{(3)} \frac{M}{n} \right), \quad (48)$$

where

$$|\delta_{s,n}^{(i)}| < 14\pi, \quad i = \overline{1,3}. \quad (49)$$

We represent the function  $P_{\alpha,r,\beta}^{(n)}(t)$ , which is defined by formula (40), in the form

$$P_{\alpha,r,\beta}^{(n)}(t) = g_{\alpha,r,n}(t) \cos\left(nt - \frac{\beta\pi}{2}\right) + h_{\alpha,r,n}(t) \sin\left(nt - \frac{\beta\pi}{2}\right), \quad (50)$$

where

$$g_{\alpha,r,n}(t) := \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} \cos kt, \quad (51)$$

$$h_{\alpha,r,n}(t) := - \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} \sin kt. \quad (52)$$

Let us show, that for functions  $g_{\alpha,r,n}$  and  $h_{\alpha,r,n}$  the following conditions are satisfied

$$\sqrt{g_{\alpha,r,n}^2(t) + h_{\alpha,r,n}^2(t)} \neq 0 \quad (53)$$

and

$$M_n = M_n(\alpha; r) := \sup_{t \in \mathbb{R}} \frac{\sqrt{(g'_{\alpha,r,n}(t))^2 + (h'_{\alpha,r,n}(t))^2}}{\sqrt{g_{\alpha,r,n}^2(t) + h_{\alpha,r,n}^2(t)}} < \infty. \quad (54)$$

Since, for arbitrary  $\alpha > 0$ ,  $0 < r < 1$  the sequence  $\{e^{-\alpha(k+n)r}\}_{k=0}^{\infty}$  is convex downwards, then (see, e.g., [23, Chapter 10, Section 2])

$$\frac{1}{2}e^{-\alpha n r} + \sum_{k=1}^{\infty} e^{-\alpha(k+n)r} \cos kt \geq 0,$$

and

$$\sqrt{g_{\alpha,r,n}^2(t) + h_{\alpha,r,n}^2(t)} \geq \frac{1}{2}e^{-\alpha n r} > 0. \quad (55)$$

Further, since

$$g'_{\alpha,r,n}(t) = - \sum_{k=1}^{\infty} k e^{-\alpha(k+n)r} \sin kt, \quad (56)$$

$$h'_{\alpha,r,n}(t) = - \sum_{k=1}^{\infty} k e^{-\alpha(k+n)r} \cos kt, \quad (57)$$

it is clear that

$$\sqrt{(g'_{\alpha,r,n}(t))^2 + (h'_{\alpha,r,n}(t))^2} < \sum_{k=1}^{\infty} k e^{-\alpha(k+n)r} < \infty. \quad (58)$$

On the basis of (55) and (58), the functions  $g_{\alpha,r,n}(t)$  and  $h_{\alpha,r,n}(t)$  satisfy the conditions (53) and (54). Therefore, setting in lemma 1  $g(t) = g_{\alpha,r,n}(t)$ ,  $h(t) = h_{\alpha,r,n}(t)$ ,  $s = p'$  and  $\gamma = -\frac{\beta\pi}{2}$ , we get that for

$$n \geq \begin{cases} 4\pi p' M_n, & 1 \leq p' < \infty, \\ 1, & p' = \infty, \end{cases} \quad (59)$$

the estimate takes place

$$\inf_{\lambda \in \mathbb{R}} \|P_{\alpha,r,\beta}^{(n)}(t) - \lambda\|_{p'} = \left\| \sqrt{(g_{\alpha,r,n}(t))^2 + (h_{\alpha,r,n}(t))^2} \right\|_{p'} \left( \frac{\|\cos t\|_{p'}}{(2\pi)^{\frac{1}{p'}}} + \delta_n^{(1)} \frac{M_n}{n} \right), \quad (60)$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ , quantity  $M_n$  is defined by equality (54), and the quantity  $\delta_n^{(1)} = \delta_n^{(1)}(\alpha, r, \beta, p)$  is such that  $|\delta_n^{(1)}| < 14\pi$ .

Setting

$$\mathcal{P}_{\alpha,r,n}(t) := g_{\alpha,r,n}(t) - ih_{\alpha,r,n}(t) = \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} e^{ikt}, \quad (61)$$

we have

$$\sqrt{(g'_{\alpha,r,n}(t))^2 + (h'_{\alpha,r,n}(t))^2} = |\mathcal{P}'_{\alpha,r,n}(t)|$$

and therefore

$$M_n = \sup_{t \in \mathbb{R}} \frac{|\mathcal{P}'_{\alpha,r,n}(t)|}{|\mathcal{P}_{\alpha,r,n}(t)|}. \quad (62)$$

Then, by virtue of the formulas (41), (42), (60) and (61), for all numbers  $n$ , which satisfy the condition (59), the estimate holds

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = \|\mathcal{P}_{\alpha,r,n}(t)\|_{p'} \left( \frac{\|\cos t\|_{p'}}{2^{\frac{1}{p'}} \pi^{1+\frac{1}{p'}}} + \delta_n^{(2)} \frac{M_n}{n} \right), \quad 1 \leq p \leq \infty, \quad (63)$$

where  $M_n$  is defined by equality (62), and for the quantity  $\delta_n^{(2)} = \delta_n^{(2)}(\alpha, r, \beta, p)$  is such that  $|\delta_n^{(2)}| < 14$ .

Since

$$|\mathcal{P}_{\alpha,r,n}(t)|^2 = \mathcal{P}_{\alpha,r,n}(t) \tilde{\mathcal{P}}_{\alpha,r,n}(t), \quad (64)$$

where

$$\tilde{\mathcal{P}}_{\alpha,r,n}(t) = g_{\alpha,r,n}(t) + ih_{\alpha,r,n}(t) = \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} e^{-ikt},$$

by expanding the product  $\mathcal{P}_{\alpha,r,n} \tilde{\mathcal{P}}_{\alpha,r,n}$  in the Fourier series (see, e.g., [23, Chapter 1, Section 23]), we get

$$\begin{aligned} \mathcal{P}_{\alpha,r,n}(t) \tilde{\mathcal{P}}_{\alpha,r,n}(t) &= \left( \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} e^{ikt} \right) \left( \sum_{k=-\infty}^0 e^{-\alpha(-k+n)r} e^{ikt} \right) = \\ &= \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} e^{-\alpha(j+n)r} e^{-\alpha(j+|k|+n)r} e^{ikt} = \\ &= \sum_{j=n}^{\infty} e^{-2\alpha j r} + 2 \sum_{k=1}^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j r} e^{-\alpha(j+k)r} \cos kt. \end{aligned} \quad (65)$$

Let convert the sum  $\sum_{j=n}^{\infty} e^{-2\alpha j r} + 2 \sum_{k=1}^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j r} e^{-\alpha(j+k)r} \cos kt$  with a help of Poisson summation formula.

**Assertion 1** [24, Chapter 2, Section 2.8]. Let continuous function  $\phi(x)$  be a function of bounded variation in the interval  $(0, \infty)$ ,  $\lim_{x \rightarrow \infty} \phi(x) = 0$  and

$$\int_0^{\infty} \phi(t) dt < \infty.$$

Then the following equality takes place

$$\sqrt{a} \left( \frac{\phi(0)}{2} + \sum_{k=1}^{\infty} \phi(ka) \right) = \sqrt{\frac{2\pi}{a}} \left( \frac{\Phi_c(0)}{2} + \sum_{k=1}^{\infty} \Phi_c\left(\frac{2\pi k}{a}\right) \right), \quad a > 0, \quad (66)$$

where  $\Phi_c(x)$  is the Fourier cosine transform of the function  $\phi(x)$  of the form

$$\Phi_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \phi(u) \cos xu du.$$

Let fix  $t \in [-\pi, \pi]$ ,  $\alpha > 0$ ,  $r \in (0, 1)$  and set

$$\phi(x) = 2 \sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha(j+x)^r} \cos xt, \quad x \geq 0$$

and  $a = 1$ . One can easily check that all conditions of the assertion 1 are satisfied, and therefore, according to (66) we obtain

$$\begin{aligned} & \sum_{j=n}^{\infty} e^{-2\alpha j^r} + 2 \sum_{k=1}^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha(j+k)^r} \cos kt = \\ & = 2 \int_0^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha(j+u)^r} \cos ut du + \\ & + 4 \sum_{k=1}^{\infty} \int_0^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha(j+u)^r} \cos ut \cos 2\pi k u du = \\ & = Q_n(t) + R_n(t), \end{aligned} \quad (67)$$

where

$$Q_n(t) = Q_n(\alpha; r; t) := 2 \sum_{j=n}^{\infty} e^{-\alpha j^r} \int_0^{\infty} e^{-\alpha(j+u)^r} \cos ut du, \quad (68)$$

$$R_n(t) = R_n(\alpha; r; t) :=$$

$$:= 2 \sum_{k=1}^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j^r} \int_0^{\infty} e^{-\alpha(j+u)^r} (\cos((t - 2\pi k)u) + \cos((t + 2\pi k)u)) du. \quad (69)$$

Hence, as a consequence of (64), (65) and (67)

$$\left| \mathcal{P}_{\alpha,r,n}(t) \right|^2 = Q_n(t) + R_n(t). \quad (70)$$

Denote by  $n_2 = n_2(\alpha, r, p)$  the smallest number  $n$  such that

$$\frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r \chi(p)}{n^{1-r}} \leq \frac{1}{14}, \quad (71)$$

where

$$\chi(p) = \begin{cases} p, & 1 \leq p < \infty, \\ 1, & p = \infty, \end{cases}$$

and let us show that for the quantity  $Q_n(t)$  for  $n \geq n_2(\alpha, r, p)$  and arbitrary  $t \in [-\pi, \pi]$  the following estimate takes place

$$Q_n(t) = \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} \left( 1 + \Theta_{\alpha,r,n}^{(4)}(t) \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \quad |\Theta_{\alpha,r,n}^{(4)}(t)| < 5. \quad (72)$$

Integrating by parts, we find

$$\begin{aligned} & \int e^{-\alpha(j+u)^r} \cos ut du = \\ & = e^{-\alpha(j+u)^r} \frac{-\alpha r (j+u)^{r-1} \cos ut + t \sin ut}{t^2 + (\alpha r (j+u)^{r-1})^2} + \alpha r (1-r) \times \\ & \times \int e^{-\alpha(j+u)^r} (j+u)^{r-2} \frac{((\alpha r (j+u)^{r-1})^2 - t^2) \cos ut - 2t \alpha r (j+u)^{r-1} \sin ut}{(t^2 + (\alpha r (j+u)^{r-1})^2)^2} du. \end{aligned}$$

Hence, we obtain the equality

$$\begin{aligned} & \int_0^\infty e^{-\alpha(j+u)^r} \cos ut du = \frac{\alpha r j^{r-1}}{t^2 + (\alpha r j^{r-1})^2} e^{-\alpha j^r} + \alpha r (1-r) \times \\ & \times \int_0^\infty e^{-\alpha(j+u)^r} (j+u)^{r-2} \frac{((\alpha r (j+u)^{r-1})^2 - t^2) \cos ut - 2t \alpha r (j+u)^{r-1} \sin ut}{(t^2 + (\alpha r (j+u)^{r-1})^2)^2} du. \end{aligned} \quad (73)$$

It is easy to verify that

$$\begin{aligned} & \left| \int_0^\infty e^{-\alpha(j+u)^r} (j+u)^{r-2} \frac{((\alpha r (j+u)^{r-1})^2 - t^2) \cos ut - 2t \alpha r (j+u)^{r-1} \sin ut}{((\alpha r (j+u)^{r-1})^2 + t^2)^2} du \right| \leq \\ & \leq \int_0^\infty e^{-\alpha(j+u)^r} (j+u)^{r-2} \left( \frac{1}{t^2 + (\alpha r (j+u)^{r-1})^2} + \frac{2t \alpha r (j+u)^{r-1}}{(t^2 + (\alpha r (j+u)^{r-1})^2)^2} \right) du \leq \\ & \leq 2 \int_0^\infty e^{-\alpha(j+u)^r} \frac{(j+u)^{r-2}}{t^2 + (\alpha r (j+u)^{r-1})^2} du. \end{aligned} \quad (74)$$

For fixed  $\alpha > 0$ ,  $r \in (0, 1)$  and  $t \in [-\pi, \pi]$  the function  $\frac{v^{r-2}}{t^2 + (\alpha r v^{r-1})^2}$ ,  $v \geq 1$  decreases. Besides, according to (29), for  $\delta = 0$ ,  $\gamma = \alpha$ ,  $m = j$ ,  $j \geq n_2(\alpha, r, p)$  the estimate takes place

$$\begin{aligned} \int_0^\infty e^{-\alpha(j+u)r} \frac{(j+u)^{r-2}}{t^2 + (\alpha r(j+u)^{r-1})^2} du &\leq \frac{j^{r-2}}{t^2 + (\alpha r j^{r-1})^2} \int_0^\infty e^{-\alpha(j+u)r} du = \\ &= \frac{j^{r-2}}{t^2 + (\alpha r j^{r-1})^2} \int_j^\infty e^{-\alpha u r} du \leq \frac{14}{13} \frac{e^{-\alpha j r}}{\alpha r j (t^2 + (\alpha r j^{r-1})^2)}. \end{aligned} \quad (75)$$

It follows from relations (73)–(75) that for  $j \geq n_2(\alpha, r, p)$

$$\begin{aligned} \int_0^\infty e^{-\alpha(j+u)r} \cos ut du &= \\ &= \frac{\alpha r j^{r-1}}{t^2 + (\alpha r j^{r-1})^2} e^{-\alpha j r} \left( 1 + \Theta_{\alpha, r, j}^{(5)}(t) \frac{1-r}{\alpha r} \frac{1}{j^r} \right), \quad |\Theta_{\alpha, r, j}^{(5)}(t)| \leq \frac{28}{13}. \end{aligned} \quad (76)$$

Therefore, taking into account (68), for  $n \geq n_2(\alpha, r, p)$  we have

$$Q_n(t) = 2\alpha r \sum_{j=n}^\infty \frac{e^{-2\alpha j r} j^{r-1}}{t^2 + (\alpha r j^{r-1})^2} \left( 1 + \Theta_{\alpha, r, n}^{(6)}(t) \frac{1-r}{\alpha r} \frac{1}{n^r} \right), \quad |\Theta_{\alpha, r, n}^{(6)}(t)| \leq \frac{28}{13}. \quad (77)$$

Further, let us find bilateral estimates for the quantities  $\sum_{j=n}^\infty \frac{e^{-2\alpha j r} j^{r-1}}{t^2 + (\alpha r j^{r-1})^2}$  for  $n \geq n_2(\alpha, r, p)$ . It can be shown that for fixed  $\alpha > 0$ ,  $r \in (0, 1)$  and  $t \in [-\pi, \pi]$  the function  $\xi(u) = \frac{e^{-2\alpha u r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2}$  decreases for  $u \geq n_2(\alpha, r, p)$ . Therefore, on basis of (30)

$$\begin{aligned} 2\alpha r \sum_{j=n}^\infty e^{-2\alpha j r} \frac{j^{r-1}}{t^2 + (\alpha r j^{r-1})^2} &= \\ &= 2\alpha r \int_n^\infty e^{-2\alpha u r} \frac{u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du + \Theta_{\alpha, r, n}^{(7)}(t) \frac{\alpha r e^{-2\alpha n r} n^{r-1}}{t^2 + (\alpha r n^{r-1})^2}, \quad 0 \leq \Theta_{\alpha, r, n}^{(7)}(t) \leq 2. \end{aligned} \quad (78)$$

Integrating by parts, we have

$$\begin{aligned} 2\alpha r \int_n^\infty \frac{e^{-2\alpha u r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du &= \\ &= \frac{e^{-2\alpha n r}}{t^2 + (\alpha r n^{r-1})^2} + 2(\alpha r)^2 (1-r) \int_n^\infty \frac{e^{-2\alpha u r} u^{2r-3}}{(t^2 + (\alpha r u^{r-1})^2)^2} du. \end{aligned} \quad (79)$$

Since

$$(\alpha r)^2 \int_n^\infty \frac{e^{-2\alpha u r} u^{2r-3}}{(t^2 + (\alpha r u^{r-1})^2)^2} du \leq \int_n^\infty \frac{e^{-2\alpha u r} u^{-1}}{t^2 + (\alpha r u^{r-1})^2} du \leq$$

$$\leq \frac{1}{n^r} \int_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du, \quad (80)$$

it follows from (79) that for  $n \geq n_2(\alpha, r, p)$  the following inequalities are true

$$\begin{aligned} & \int_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du \leq \\ & \leq \frac{1}{2\alpha r} \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} + \frac{1-r}{\alpha r} \frac{1}{n^r} \int_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du \leq \\ & \leq \frac{1}{2\alpha r} \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} + \frac{1}{14} \int_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du. \end{aligned}$$

Hence, for  $n \geq n_2(\alpha, r, p)$

$$\int_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du \leq \frac{7}{13\alpha r} \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2}. \quad (81)$$

From (79)–(81) for  $n \geq n_2(\alpha, r, p)$  we arrive at the following estimate

$$\begin{aligned} & 2\alpha r \int_n^\infty e^{-2\alpha u^r} \frac{u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du = \\ & = \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} \left( 1 + \Theta_{\alpha, r, n}^{(8)}(t) \frac{1-r}{\alpha r} \frac{1}{n^r} \right), \quad 0 < \Theta_{\alpha, r, n}^{(8)}(t) \leq \frac{14}{13}. \end{aligned} \quad (82)$$

It follows from formulas (78) and (82) that

$$\begin{aligned} & 2\alpha r \sum_{j=n}^\infty \frac{e^{-2\alpha j^r} j^{r-1}}{t^2 + (\alpha r j^{r-1})^2} = \\ & = \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} \left( 1 + \Theta_{\alpha, r, n}^{(9)}(t) \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \end{aligned} \quad (83)$$

where  $n \geq n_2(\alpha, r, p)$  and  $0 < \Theta_{\alpha, r, n}^{(9)}(t) \leq 2$ .

In view of (77) and (83) for all  $n \geq n_2(\alpha, r, p)$  we obtain (72). In particular, it follows from formulas (71) and (72) that

$$Q_n(t) > 0, \quad t \in [-\pi, \pi], \quad n \geq n_2(\alpha, r, p). \quad (84)$$

Let us find upper estimate for the quantity  $R_n(t)$  of the form (69). Denote by  $\mathfrak{M}$  the set of all convex downwards, continuous functions  $\psi(t) > 0$ ,  $t \geq 1$ , such that  $\lim_{t \rightarrow \infty} \psi(t) = 0$ . The following assertion takes place.



**Lemma 2.** *Let  $\psi \in \mathfrak{M}$ . Then*

$$0 < \int_0^\infty \psi(\tau + u) \cos vu du \leq \frac{\pi}{v^2} |\psi'(\tau)|, \quad v \in \mathbb{R} \setminus \{0\}, \quad \tau \geq 1. \quad (85)$$

**Proof of lemma 2.** We use the scheme of the proof of the estimate (2.4.31) from the work [25, p. 93]. Let, e.g., consider the case  $v > 0$ . Using the method of integration by parts, we have

$$\int_0^\infty \psi(\tau + u) \cos vu du = \frac{-1}{v} \int_0^\infty \psi'(\tau + u) \sin vu du. \quad (86)$$

We set

$$I(x) = I(\psi; \tau; v; x) := - \int_x^\infty \psi'(j + u) \sin vu du, \quad x \geq 0, \quad v > 0, \quad \tau \in \mathbb{N}.$$

The function  $I(x)$ , obviously, is continuous for every fixed  $v$ , and on every interval between the consecutive zeros  $u_m = \frac{\pi m}{v}$  and  $u_{m+1} = \frac{\pi(m+1)}{v}$  of the function  $\sin vu$  has one simple zero  $x_m$ . Existence of zeros  $x_m$  of the function  $I(x)$  is a consequence of the Leibniz theorem on alternating series, and uniqueness of zero  $x_m$  on the interval  $(u_m, u_{m+1})$  follows from the equality

$$\text{sign } I'(x) = -\text{sign } \sin xv, \quad x \in (u_m, u_{m+1}) \quad m \in \mathbb{Z}_+.$$

Let  $x_0$  be the zero closest from the right to the point  $x = 0$ . It is obvious that

$$0 \leq x_0 \leq \frac{\pi}{v}.$$

Taking into account this fact and also monotone decreasing of the function  $-\psi'(t)$  on the interval  $[1, \infty)$ , we have

$$\begin{aligned} \frac{-1}{v} \int_0^\infty \psi'(\tau + u) \sin vu du &= \frac{1}{v} \int_0^{x_0} |\psi'(\tau + u)| \sin vu du \leq \\ &\leq \frac{1}{v} \int_0^{\frac{\pi}{v}} |\psi'(\tau + u)| du \leq \frac{\pi}{v^2} |\psi'(\tau)|. \end{aligned} \quad (87)$$

For  $v > 0$  inequality (85) follows from the formulas (86) and (87). For  $v < 0$  the proof of inequality (85) is analogous. Lemma 2 is proved.

Setting in inequality (85)  $v = t \pm 2\pi k$ ,  $k \in \mathbb{N}$ , and  $\tau = j$ , we obtain that for arbitrary  $\psi \in \mathfrak{M}$  and  $t \in [-\pi, \pi]$

$$0 < \sum_{k=1}^\infty \sum_{j=n}^\infty \psi(j) \int_0^\infty \psi(j + u) (\cos((t - 2\pi k)u) + \cos((t + 2\pi k)u)) du \leq$$

$$\begin{aligned}
&\leq \pi \sum_{k=1}^{\infty} \left( \frac{1}{(t-2k\pi)^2} + \frac{1}{(t+2k\pi)^2} \right) \sum_{j=n}^{\infty} \psi(j) |\psi'(j)| \leq \\
&\leq \pi \sum_{k=1}^{\infty} \left( \frac{1}{(\pi-2k\pi)^2} + \frac{1}{(\pi+2k\pi)^2} \right) \psi(n) \left( |\psi'(n)| + \int_n^{\infty} |\psi'(u)| du \right) = \\
&= \frac{1}{\pi} \sum_{k=1}^{\infty} \left( \frac{1}{(2k-1)^2} + \frac{1}{(2k+1)^2} \right) \psi(n) \left( |\psi'(n)| + \psi(n) \right) = \\
&= \left( \frac{\pi}{4} - \frac{1}{\pi} \right) \psi(n) \left( |\psi'(n)| + \psi(n) \right), \quad n \in \mathbb{N}.
\end{aligned} \tag{88}$$

Setting in (88)  $\psi(t) = e^{-\alpha t^r}$ ,  $0 < r < 1$ ,  $\alpha > 0$ , we get that for the function  $R_n(t)$  of the form (69) the following estimate takes place

$$0 < R_n(t) \leq \left( \frac{\pi}{2} - \frac{2}{\pi} \right) e^{-2\alpha n^r} \left( \frac{\alpha r}{n^{1-r}} + 1 \right) \leq \left( \frac{\pi}{2} - \frac{2}{\pi} \right) \frac{15}{14} e^{-2\alpha n^r} < \frac{\pi}{3} e^{-2\alpha n^r}, \tag{89}$$

where  $n \geq n_2(\alpha, r, p)$ .

By virtue of (70)

$$|\mathcal{P}_{\alpha, r, n}(t)| = \sqrt{Q_n(t) + R_n(t)}, \tag{90}$$

and therefore, taking into account (84) and (89), we have

$$\|\mathcal{P}_{\alpha, r, n}\|_{p'} = \|\sqrt{Q_n}\|_{L_{p'}[-\pi, \pi]} + \Theta_{\alpha, r, p, n}^{(2)} e^{-\alpha n^r}, \quad 1 \leq p' \leq \infty, \tag{91}$$

where  $|\Theta_{\alpha, r, p, n}^{(2)}| < \frac{2\pi^2}{3}$  and  $n \geq n_2(\alpha, r, p)$ .

Let us show, that for  $1 \leq p' \leq \infty$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ , and  $n \geq n_2(\alpha, r, p)$  the estimate is true

$$\begin{aligned}
\|\mathcal{P}_{\alpha, r, n}\|_{p'} &= e^{-\alpha n^r} n^{\frac{1-r}{p}} \left( \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \right. \\
&\quad \left. + \Theta_{\alpha, r, p, n}^{(3)} \left( \frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right) \right),
\end{aligned} \tag{92}$$

where

$$|\Theta_{\alpha, r, p, n}^{(3)}| \leq \begin{cases} \pi^2, & 1 \leq p' < \infty, \\ \frac{14}{13}, & p' = \infty. \end{cases} \tag{93}$$

Since, on the basis of estimate (72) for  $n \geq n_2(\alpha, r, p)$  and  $1 \leq p' \leq \infty$

$$\begin{aligned}
&\left| \left( 1 + \Theta_{\alpha, r, n}^{(10)}(t) \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right)^{\frac{1}{2}} - 1 \right) \right| \leq \\
&\leq \frac{5}{2} \frac{1}{\sqrt{1 - 5 \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right)}} \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \leq \frac{5\sqrt{7}}{3\sqrt{2}} \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right)
\end{aligned}$$

we get

$$\sqrt{Q_n(t)} = \frac{e^{-\alpha n^r}}{\sqrt{t^2 + (\alpha r n^{r-1})^2}} \left( 1 + \Theta_{\alpha,r,n}^{(10)}(t) \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \quad |\Theta_{\alpha,r,n}^{(10)}(t)| \leq \frac{5\sqrt{7}}{3\sqrt{2}}. \quad (94)$$

For  $1 \leq p' < \infty$  from (94) we have

$$\begin{aligned} \|\sqrt{Q_n}\|_{L_{p'}[-\pi,\pi]} &= e^{-\alpha n^r} \left( \int_{-\pi}^{\pi} \frac{dt}{(t^2 + (\alpha r n^{r-1})^2)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} \left( 1 + \Theta_{\alpha,r,p,n}^{(4)} \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right) = \\ &= 2^{\frac{1}{p'}} e^{-\alpha n^r} \left( \frac{n^{1-r}}{\alpha r} \right)^{\frac{1}{p}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \left( 1 + \Theta_{\alpha,r,p,n}^{(4)} \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \end{aligned} \quad (95)$$

where  $|\Theta_{\alpha,r,p,n}^{(4)}| \leq \frac{5\sqrt{7}}{3\sqrt{2}}$  and  $J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right)$  is defined by equality (13).

Combining (91) and (95), we obtain that for  $1 \leq p' < \infty$  the following relation takes place

$$\begin{aligned} \|\mathcal{P}_{\alpha,r,n}\|_{p'} &= e^{-\alpha n^r} n^{\frac{1-r}{p}} \left( \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \right. \\ &\quad \left. + \Theta_{\alpha,r,p,n}^{(4)} \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) + \frac{\Theta_{\alpha,r,p,n}^{(2)}}{n^{\frac{1-r}{p}}} \right). \end{aligned} \quad (96)$$

However, for all  $n > n_2(\alpha, r, p)$

$$\frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{\alpha r}{n^{1-r}} < \frac{1}{n^{\frac{1-r}{p}}}, \quad 1 \leq p' < \infty. \quad (97)$$

Indeed, taking into account (13) and (71), for all  $1 < p' < \infty$  and  $n \geq n_2(\alpha, r, p)$  we find

$$\begin{aligned} \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{\alpha r}{n^{1-r}} n^{\frac{1-r}{p}} &= \left( \frac{2\alpha r}{n^{1-r}} \right)^{\frac{1}{p'}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) < \\ &< \left( \frac{2\alpha r}{n^{1-r}} \right)^{\frac{1}{p'}} \left( \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} \right)^{\frac{1}{p'}} < \left( \frac{2\alpha r}{n^{1-r}} \right)^{\frac{1}{p'}} \left( 1 + \int_1^\infty \frac{dt}{t^{p'}} \right)^{\frac{1}{p'}} = \\ &= \left( \frac{2\alpha r p}{n^{1-r}} \right)^{\frac{1}{p'}} < \left( \frac{1}{7} \right)^{\frac{1}{p'}} < 1, \end{aligned} \quad (98)$$

and for  $p' = 1$  and  $n \geq n_2(\alpha, r, p)$ , taking into account decreasing on the interval  $[e, \infty)$  of the function  $\frac{\ln v}{v}$ , we have

$$\begin{aligned} \frac{2\alpha r}{n^{1-r}} J_1 \left( \frac{\pi n^{1-r}}{\alpha r} \right) &= \frac{2\alpha r}{n^{1-r}} \int_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2 + 1}} < \frac{2\alpha r}{n^{1-r}} \left( 1 + \int_1^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2 + 1}} \right) < \\ &< \frac{2\alpha r}{n^{1-r}} + \frac{2\alpha r}{n^{1-r}} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) \leq \frac{2}{14} + \frac{2\pi \ln 14\pi}{14\pi} < 1. \end{aligned} \quad (99)$$

Formulas (98) and (99) prove (97). For  $1 \leq p' < \infty$  estimate (92) follows from (96) and (97).

Let us verify validity of the estimate (92) for  $p' = \infty$ . It follows from (61) and (30) that

$$\|\mathcal{P}_{\alpha,r,n}\|_{\infty} = \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} = \int_n^{\infty} e^{-\alpha t r} dt + \Theta_{\alpha,r,n}^{(11)} e^{-\alpha n r}, \quad |\Theta_{\alpha,r,n}^{(11)}| \leq 1. \quad (100)$$

Setting in formula (26)  $\gamma = \alpha$ ,  $\delta = 0$  and  $m = n$ , from (100) we obtain that for arbitrary  $n \geq n_2(\alpha, r, p)$

$$\|\mathcal{P}_{\alpha,r,n}\|_{\infty} = \frac{e^{-\alpha n r}}{\alpha r} n^{1-r} \left( 1 + \Theta_{\alpha,r,n}^{(12)} \left( \frac{1-r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \quad (101)$$

where  $|\Theta_{\alpha,r,n}^{(12)}| \leq \frac{14}{13}$ .

For  $p' = \infty$  the validity of (92) follows from (101) and the equality  $J_{\infty}(\frac{\pi n^{1-r}}{\alpha r}) = 1$ .

To complete the proof of theorem 1 it suffices to find the upper estimate of the quantity  $M_n$  in formula (63). It is clear that

$$\begin{aligned} M_n &= \sup_{t \in \mathbb{R}} \frac{|\mathcal{P}'_{\alpha,r,n}(t)| |\mathcal{P}_{\alpha,r,n}(t)|}{|\mathcal{P}_{\alpha,r,n}(t)|^2} = \\ &= \max \left\{ \sup_{|t| \leq \frac{\alpha r}{n^{1-r}}} \frac{|\mathcal{P}'_{\alpha,r,n}(t)| |\mathcal{P}_{\alpha,r,n}(t)|}{|\mathcal{P}_{\alpha,r,n}(t)|^2}, \sup_{\frac{\alpha r}{n^{1-r}} \leq |t| \leq \pi} \frac{|\mathcal{P}'_{\alpha,r,n}(t)| |\mathcal{P}_{\alpha,r,n}(t)|}{|\mathcal{P}_{\alpha,r,n}(t)|^2} \right\}. \end{aligned} \quad (102)$$

In view of formulas (71) and (72) and the fact that  $R_n(t) > 0$  for  $n \geq n_2(\alpha, r, p)$  we obtain

$$|\mathcal{P}_{\alpha,r,n}(t)|^2 > Q_n(t) > \frac{9}{14} \frac{e^{-2\alpha n r}}{t^2 + (\alpha r n^{r-1})^2}. \quad (103)$$

It directly follows from (61) that

$$|\mathcal{P}_{\alpha,r,n}(t)| \leq \sum_{k=0}^{\infty} e^{-\alpha(k+n)r}, \quad |\mathcal{P}'_{\alpha,r,n}(t)| \leq \sum_{k=1}^{\infty} k e^{-\alpha(k+n)r}. \quad (104)$$

By virtue of (101) for  $n \geq n_2(\alpha, r, p)$  we have

$$|\mathcal{P}_{\alpha,r,n}(t)| \leq \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} < \frac{14}{13} e^{-\alpha n r} \frac{n^{1-r}}{\alpha r}. \quad (105)$$

The function  $t e^{-\alpha t r}$  is monotone decreasing for  $t > (\alpha r)^{-\frac{1}{r}}$ . Therefore, according to (30), for  $n \geq n_2(\alpha, r, p)$  the following estimate takes place

$$\begin{aligned} \sum_{k=1}^{\infty} e^{-\alpha(k+n)r} k &= \sum_{k=n}^{\infty} e^{-\alpha k r} k - n \sum_{k=n}^{\infty} e^{-\alpha k r} \leq \\ &\leq e^{-\alpha n r} n + \int_n^{\infty} e^{-\alpha t r} t dt - n \int_n^{\infty} e^{-\alpha t r} dt. \end{aligned} \quad (106)$$

Setting in (26)  $\gamma = \alpha$ ,  $\delta = 1$ ,  $m = n$ , and also  $\gamma = \alpha$ ,  $\delta = 0$ ,  $m = n$ , from (104) and (106) we have

$$|\mathcal{P}'_{\alpha,r,n}(t)| \leq e^{-\alpha n^r} \left( \frac{42}{13} \left( \frac{n^{1-r}}{\alpha r} \right)^2 + n \right), \quad n \geq n_2(\alpha, r, p). \quad (107)$$

In view of (103), (105) and (107) for  $n \geq n_2(\alpha, r, p)$  we arrive at the estimate

$$\begin{aligned} & \sup_{|t| \leq \frac{\alpha r}{n^{1-r}}} \frac{|\mathcal{P}'_{\alpha,r,n}(t)| |\mathcal{P}_{\alpha,r,n}(t)|}{|\mathcal{P}_{\alpha,r,n}(t)|^2} \leq \\ & \leq \frac{14}{9} e^{2\alpha n^r} \sup_{|t| < \frac{\alpha r}{n^{1-r}}} |\mathcal{P}'_{\alpha,r,n}(t)| |\mathcal{P}_{\alpha,r,n}(t)| (t^2 + \left( \frac{\alpha r}{n^{1-r}} \right)^2) \leq \\ & \leq \frac{5488}{507} \left( \left( \frac{n^{1-r}}{\alpha r} \right)^2 + n \right) \frac{n^{1-r}}{\alpha r} \left( \frac{\alpha r}{n^{1-r}} \right)^2 = \frac{5488}{507} \left( \frac{n^{1-r}}{\alpha r} + \alpha r n^r \right). \end{aligned} \quad (108)$$

Applying the Abel transformation to the function  $\mathcal{P}_{\alpha,r,n}(t)$  for  $0 < |t| \leq \pi$ , and taking into account the inequality

$$\left| \sum_{j=0}^k e^{ijt} \right| \leq \frac{\pi}{|t|}, \quad 0 < |t| \leq \pi,$$

we get

$$|\mathcal{P}_{\alpha,r,n}(t)| = \left| \sum_{k=0}^{\infty} (e^{-\alpha(k+n)^r} - e^{-\alpha(k+n+1)^r}) \sum_{j=0}^k e^{ijt} \right| \leq \frac{\pi}{|t|} e^{-\alpha n^r}. \quad (109)$$

By analogy, for  $0 < |t| \leq \pi$

$$\begin{aligned} |\mathcal{P}'_{\alpha,r,n}(t)| &= \left| \sum_{k=0}^{\infty} (e^{-\alpha(k+n)^r} k - e^{-\alpha(k+n+1)^r} (k+1)) \sum_{j=0}^k e^{ijt} \right| \leq \\ &\leq \frac{\pi}{|t|} \sum_{k=0}^{\infty} |e^{-\alpha(k+n)^r} k - e^{-\alpha(k+n+1)^r} (k+1)| \leq \\ &\leq \frac{\pi}{|t|} \left( \sum_{k=0}^{\infty} k (e^{-\alpha(k+n)^r} - e^{-\alpha(k+n+1)^r}) + \sum_{k=0}^{\infty} e^{-\alpha(k+n+1)^r} \right) = \end{aligned} \quad (110)$$

According to (105) and (110)

$$|\mathcal{P}'_{\alpha,r,n}(t)| \leq \frac{2\pi}{|t|} \sum_{k=0}^{\infty} e^{-\alpha(k+n+1)^r} \leq \frac{28\pi}{13|t|} e^{-\alpha n^r} \frac{n^{1-r}}{\alpha r}. \quad (111)$$

In view of (103), (109) and (111) we obtain the estimate

$$\sup_{\frac{\alpha r}{n^{1-r}} \leq |t| \leq \pi} \frac{|\mathcal{P}'_{\alpha,r,n}(t)| |\mathcal{P}_{\alpha,r,n}(t)|}{|\mathcal{P}_{\alpha,r,n}(t)|^2} \leq$$

$$\begin{aligned}
&\leq \frac{14}{9} e^{2\alpha n^r} \sup_{\frac{\alpha r}{n^{1-r}} \leq |t| \leq \pi} |\mathcal{P}'_{\alpha,r,n}(t)| |\mathcal{P}_{\alpha,r,n}(t)| \left( t^2 + \left( \frac{\alpha r}{n^{1-r}} \right)^2 \right) \leq \\
&\leq \frac{392\pi^2}{117} \frac{n^{1-r}}{\alpha r} \sup_{\frac{\alpha r}{n^{1-r}} \leq |t| \leq \pi} \frac{t^2 + \left( \frac{\alpha r}{n^{1-r}} \right)^2}{t^2} \leq \frac{784\pi^2}{117} \frac{n^{1-r}}{\alpha r}.
\end{aligned} \tag{112}$$

Combining (102), (108) and (112), we arrive at the estimate

$$M_n \leq \frac{784\pi^2}{117} \left( \frac{n^{1-r}}{\alpha r} + \alpha r n^r \right), \quad n \geq n_2(\alpha, r, p). \tag{113}$$

It follows from conditions (14) and (71) that  $n_0(\alpha, r, p) \geq n_2(\alpha, r, p)$  for arbitrary  $1 \leq p \leq \infty$ . It means that estimates (92) and (113) are true also for  $n \geq n_0(\alpha, r, p)$ . Let us show that for  $n \geq n_0(\alpha, r, p)$  the condition (59) is satisfied. This is obvious for  $p' = \infty$ . For  $1 \leq p' < \infty$  by virtue of (113), we have

$$4\pi M_n p' \leq \frac{3136\pi^3}{117} \left( \frac{n^{1-r}}{\alpha r} + \alpha r n^r \right) p' < 27\pi^3 \left( \frac{n^{1-r}}{\alpha r} + \alpha r \chi(p) n^r \right) p'. \tag{114}$$

According to (14) and (114) for any  $n \geq n_0(\alpha, r, p)$  the following inequality is true

$$4\pi p' M_n \leq n,$$

which is equivalent to (59) for  $1 \leq p' < \infty$ .

By using formulas (63), (92) and (113) for  $n \geq n_0(\alpha, r, p)$  we arrive at the estimate

$$\begin{aligned}
&\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = \\
&= e^{-\alpha n^r} n^{\frac{1-r}{p}} \left( \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \Theta_{\alpha,r,p,n}^{(3)} \left( \frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right) \right) \times \\
&\quad \times \left( \frac{\|\cos t\|_{p'}}{2^{\frac{1}{p'}} \pi^{1+\frac{1}{p'}}} + \delta_n^{(3)} \left( \frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \quad 1 \leq p \leq \infty,
\end{aligned} \tag{115}$$

where for  $\Theta_{\alpha,r,p,n}^{(3)}$  the estimate (93) takes place, and  $|\delta_n^{(3)}| < \frac{10976\pi^2}{117}$ .

For  $n \geq n_0(\alpha, r, p)$  the following inequality holds

$$\begin{aligned}
&|\delta_n^{(3)}| \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \left( \frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) < \\
&< \frac{21952\pi^2}{117} \left( \frac{1}{(\alpha r)^{1+\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right),
\end{aligned} \tag{116}$$

which follows from (97) for  $1 \leq p' < \infty$ , and it is obvious for  $p' = \infty$ . Besides, according to (93) and (14) for  $n \geq n_0(\alpha, r, p)$

$$\left| \Theta_{\alpha,r,p,n}^{(3)} \right| \left( \frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right) \left( \frac{\|\cos t\|_{p'}}{2^{\frac{1}{p'}} \pi^{1+\frac{1}{p'}}} + |\delta_n^{(3)}| \left( \frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right) <$$

$$< \frac{363\pi^2}{50} \left( \frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}} \right). \quad (117)$$

In view of formulas (115)–(117) we arrive at (15). Theorem 1 is proved.

**4. Proof of lemma 1.** It is obvious that for  $1 \leq s \leq \infty$

$$\inf_{\lambda \in \mathbb{R}} \|\phi(t) - \lambda\|_s \leq \|\phi\|_s,$$

$$\frac{1}{2} \|\phi(t + \frac{\pi}{n}) - \phi(t)\|_s \leq \sup_{h \in \mathbb{R}} \frac{1}{2} \|\phi(t + h) - \phi(t)\|_s$$

and

$$\sup_{h \in \mathbb{R}} \frac{1}{2} \|\phi(t + h) - \phi(t)\|_s \leq \inf_{\lambda \in \mathbb{R}} \|\phi(t) - \lambda\|_s.$$

Hence, in order to proof lemma it suffices to verify the validity of formula (46) and relation

$$\frac{1}{2} \|\phi(t + \frac{\pi}{n}) - \phi(t)\|_s \geq \|r\|_s \left( \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} - 14\pi \frac{M}{n} \right). \quad (118)$$

First, we consider the case  $1 \leq s < \infty$ . Let verify the validity of equality (46). Setting

$$\phi_k(t) = g\left(\frac{k\pi}{n}\right) \cos(nt + \gamma) + h\left(\frac{k\pi}{n}\right) \sin(nt + \gamma), \quad k = \overline{-n+1, n}, \quad (119)$$

we get

$$\begin{aligned} \|\phi\|_s &= \left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi(t)|^s dt \right)^{\frac{1}{s}} = \\ &= \left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi_k(t)|^s dt \right)^{\frac{1}{s}} + \Theta_n^{(1)} \left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi(t) - \phi_k(t)|^s dt \right)^{\frac{1}{s}}, \quad |\Theta_n^{(1)}| \leq 1. \end{aligned} \quad (120)$$

Let us find the estimate of first term in (120). It is obvious, that according to (119)

$$\begin{aligned} &\left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi_k(t)|^s dt \right)^{\frac{1}{s}} = \\ &= \left( \sum_{k=-n+1}^n r^s \left( \frac{k\pi}{n} \right) \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \left| \cos \left( nt + \gamma - \arg \left( g\left(\frac{k\pi}{n}\right) + ih\left(\frac{k\pi}{n}\right) \right) \right) \right|^s dt \right)^{\frac{1}{s}} = \end{aligned}$$

$$= \left( \sum_{k=-n+1}^n r^s \left( \frac{k\pi}{n} \right) \frac{1}{n} \int_0^\pi |\cos t|^s dt \right)^{\frac{1}{s}} = \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} \left( \sum_{k=-n+1}^n r^s \left( \frac{k\pi}{n} \right) \frac{\pi}{n} \right)^{\frac{1}{s}}, \quad (121)$$

where  $r(t)$  is defined by formula (43), and  $i$  is imaginary unit.

Let us show that for any collection of points  $\xi_k$ ,  $k = \overline{-n+1, n}$ , such that  $\frac{(k-1)\pi}{n} \leq \xi_k \leq \frac{k\pi}{n}$ , for  $n \geq 4\pi s M$  the following estimate is true

$$\left( \sum_{k=-n+1}^n r^s(\xi_k) \frac{\pi}{n} \right)^{\frac{1}{s}} = \|r\|_s \left( 1 + \Theta_n^{(2)} \frac{M}{n} \right), \quad |\Theta_n^{(2)}| \leq 4. \quad (122)$$

Indeed, since

$$\sum_{k=-n+1}^n r^s(\xi_k) \frac{\pi}{n} = \int_{-\pi}^\pi r^s(t) dt + \Theta_n^{(3)} \frac{\bigvee^\pi(r^s)}{n}, \quad |\Theta_n^{(3)}| \leq \pi,$$

and under the condition

$$n \geq \frac{2\pi \bigvee^\pi(r^s)}{\|r\|_s^s} \quad (123)$$

$$\left( \int_{-\pi}^\pi r^s(t) dt + \Theta_n^{(3)} \frac{\bigvee^\pi(r^s)}{n} \right)^{\frac{1}{s}} = \|r\|_s \left( 1 + \Theta_n^{(4)} \frac{\bigvee^\pi(r^s)}{ns\|r\|_s^s} \right), \quad |\Theta_n^{(4)}| \leq 2, \quad (124)$$

hence

$$\left( \sum_{k=-n+1}^n r^s(\xi_k) \frac{\pi}{n} \right)^{\frac{1}{s}} = \|r\|_s \left( 1 + \Theta_n^{(4)} \frac{\bigvee^\pi(r^s)}{ns\|r\|_s^s} \right), \quad |\Theta_n^{(4)}| \leq 2. \quad (125)$$

It is easy to verify that

$$\bigvee_{-\pi}^\pi(r^s) = s \int_{-\pi}^\pi r^{s-1}(t) |r'(t)| dt \leq s \|r\|_s^s \left\| \frac{r'(t)}{r(t)} \right\|_\infty, \quad (126)$$

$$\left| \frac{r'(t)}{r(t)} \right| = \left| \frac{g(t)g'(t) + h(t)h'(t)}{r^2(t)} \right| \leq \frac{|g'(t)| + |h'(t)|}{r(t)} \leq 2M, \quad t \in \mathbb{R}, \quad (127)$$

therefore

$$\frac{\bigvee^\pi(r^s)}{\|r\|_s^s} \leq s \left\| \frac{r'(t)}{r(t)} \right\|_\infty \leq 2sM. \quad (128)$$

By virtue of (128), for  $n \geq 4\pi s M$  the condition (123) is satisfied. Therefore, according to (125), the estimate (122) takes place. Setting in (122)  $\xi_k = \frac{k\pi}{n}$ ,  $k = \overline{-n+1, n}$ , in view of



(121) we obtain

$$\left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi_k(t)|^s dt \right)^{\frac{1}{s}} = \|r\|_s \left( \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} + \Theta_n^{(5)} \frac{M}{n} \right), \quad |\Theta_n^{(5)}| \leq 4. \quad (129)$$

Let us find upper estimate of the second term in (120). On the basis of (45) and (119)

$$\begin{aligned} \phi(t) - \phi_k(t) &= \\ &= \left( r(t) - r\left(\frac{k\pi}{n}\right) \right) \left( \frac{g\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \cos(nt + \gamma) + \frac{h\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \sin(nt + \gamma) \right) + \\ &+ r(t) \left( \left( \frac{g(t)}{r(t)} - \frac{g\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \right) \cos(nt + \gamma) + \left( \frac{h(t)}{r(t)} - \frac{h\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \right) \sin(nt + \gamma) \right), \end{aligned} \quad (130)$$

therefore

$$\left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi(t) - \phi_k(t)|^s dt \right)^{\frac{1}{s}} \leq I_n^{(1)} + I_n^{(2)}, \quad (131)$$

where

$$\begin{aligned} I_n^{(1)} &:= \left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \left| r(t) - r\left(\frac{k\pi}{n}\right) \right|^s (|\cos(nt + \gamma)| + |\sin(nt + \gamma)|)^s dt \right)^{\frac{1}{s}}, \\ I_n^{(2)} &:= \left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} r^s(t) \left( \left| \frac{g(t)}{r(t)} - \frac{g\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \right| |\cos(nt + \gamma)| + \right. \right. \\ &\quad \left. \left. + \left| \frac{h(t)}{r(t)} - \frac{h\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \right| |\sin(nt + \gamma)| \right)^s dt \right)^{\frac{1}{s}}. \end{aligned}$$

Using obvious inequality

$$|\cos t| + |\sin t| \leq \sqrt{2}, \quad (132)$$

Lagrange theorem and relation (127), we have

$$\begin{aligned} I_n^{(1)} &\leq \sqrt{2} \left( \sum_{k=-n+1}^n \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| r(t) - r\left(\frac{k\pi}{n}\right) \right|^s \frac{\pi}{n} \right)^{\frac{1}{s}} \leq \\ &\leq \frac{\sqrt{2}\pi}{n} \sup_{t \in \mathbb{R}} \left| \frac{r'(t)}{r(t)} \right| \left( \sum_{k=-n+1}^n \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r^s(t) \frac{\pi}{n} \right)^{\frac{1}{s}}. \end{aligned} \quad (133)$$

It follows from (122), (127) and (133), that for  $n \geq 4\pi sM$

$$I_n^{(1)} \leq 2\sqrt{2}\pi \frac{M}{n} (1 + 4\frac{M}{n}) \|r\|_s \leq 2\sqrt{2}\pi \frac{M}{n} \left(1 + \frac{1}{\pi}\right) \|r\|_s = \frac{2\sqrt{2}M(1+\pi)}{n} \|r\|_s. \quad (134)$$

It is easy to see that

$$\begin{aligned} I_n^{(2)} \leq & \left( \sum_{k=-n+1}^n \left( \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left\{ \left| \frac{g(t)}{r(t)} - \frac{g(\frac{k\pi}{n})}{r(\frac{k\pi}{n})} \right| |\cos(nt + \gamma)| + \right. \right. \right. \\ & \left. \left. \left. + \left| \frac{h(t)}{r(t)} - \frac{h(\frac{k\pi}{n})}{r(\frac{k\pi}{n})} \right| |\sin(nt + \gamma)| \right\} \right)^s \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} r^s(t) dt \right)^{\frac{1}{s}}. \end{aligned} \quad (135)$$

For any  $t_1, t_2 \in \mathbb{R}$  such that  $|t_1 - t_2| \leq \frac{\pi}{n}$  the following inequalities take place

$$\left| \frac{g(t_1)}{r(t_1)} - \frac{g(t_2)}{r(t_2)} \right| \leq \frac{3\pi M}{n}, \quad (136)$$

$$\left| \frac{h(t_1)}{r(t_1)} - \frac{h(t_2)}{r(t_2)} \right| \leq \frac{3\pi M}{n}. \quad (137)$$

Indeed, by virtue of Lagrange theorem, taking into account (44) and (127), we have

$$\begin{aligned} \left| \frac{g(t_1)}{r(t_1)} - \frac{g(t_2)}{r(t_2)} \right| & \leq \frac{\pi}{n} \sup_{t \in \mathbb{R}} \left| \frac{g'(t)r(t) - g(t)r'(t)}{r^2(t)} \right| \leq \\ & \leq \frac{\pi}{n} \sup_{t \in \mathbb{R}} \frac{|g'(t)|}{r(t)} + \frac{\pi}{n} \sup_{t \in \mathbb{R}} \frac{|r'(t)|}{r(t)} \leq \frac{3\pi M}{n}. \end{aligned} \quad (138)$$

By analogy, we prove the inequality (137). In view of (132), (136), (137) and (135) we obtain

$$I_n^{(2)} \leq \frac{3\sqrt{2}\pi M}{n} \|r\|_s, \quad n \in \mathbb{N}. \quad (139)$$

Combining (131), (134) and (139), we arrive at the estimate

$$\left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi(t) - \phi_k(t)|^s dt \right)^{\frac{1}{s}} \leq \sqrt{2}(5\pi + 2) \|r\|_s \frac{M}{n}, \quad n \geq 4\pi sM. \quad (140)$$

By comparing estimates (120), (129) and (140) we conclude that for  $n \geq 4\pi sM$

$$\|\phi\|_s = \|r\|_s \left( \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} + \delta_{s,n}^{(1)} \frac{M}{n} \right), \quad |\delta_{s,n}^{(1)}| \leq \sqrt{2}(5\pi + 2) + 4, \quad 1 \leq s < \infty. \quad (141)$$

Further, we prove the relation (118) for  $1 \leq s < \infty$ . In view of definition (45)

$$|\phi(t + \frac{\pi}{n}) - \phi(t)| =$$

$$\begin{aligned}
&= \left| 2\phi(t) + g\left(t + \frac{\pi}{n}\right) \cos(nt + \gamma) + h\left(t + \frac{\pi}{n}\right) \sin(nt + \gamma) - \right. \\
&\quad \left. - (g(t) \cos(nt + \gamma) + h(t) \sin(nt + \gamma)) \right| = \\
&= \left| 2\phi(t) + \left(r\left(t + \frac{\pi}{n}\right) - r(t)\right) \left(\frac{g\left(t + \frac{\pi}{n}\right)}{r\left(t + \frac{\pi}{n}\right)} \cos(nt + \gamma) + \frac{h\left(t + \frac{\pi}{n}\right)}{r\left(t + \frac{\pi}{n}\right)} \sin(nt + \gamma)\right) + \right. \\
&\quad \left. + r(t) \left(\left(\frac{g\left(t + \frac{\pi}{n}\right)}{r\left(t + \frac{\pi}{n}\right)} - \frac{g(t)}{r(t)}\right) \cos(nt + \gamma) + \left(\frac{h\left(t + \frac{\pi}{n}\right)}{r\left(t + \frac{\pi}{n}\right)} - \frac{h(t)}{r(t)}\right) \sin(nt + \gamma)\right) \right|, \quad (142)
\end{aligned}$$

therefore for any  $1 \leq s \leq \infty$  by virtue of (132), (136) and (137), we get

$$\begin{aligned}
&\frac{1}{2} \|\phi\left(t + \frac{\pi}{n}\right) - \phi(t)\|_s \geq \\
&\geq \|\phi\|_s - \frac{1}{\sqrt{2}} \left( \left\| r\left(t + \frac{\pi}{n}\right) - r(t) \right\|_s + 3\pi \|r\|_s \frac{M}{n} \right). \quad (143)
\end{aligned}$$

By applying the Lagrange theorem, we obtain

$$\begin{aligned}
\left\| r\left(t + \frac{\pi}{n}\right) - r(t) \right\|_s &= \left( \sum_{k=-n+1}^n \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \left| r\left(t + \frac{\pi}{n}\right) - r(t) \right|^s dt \right)^{\frac{1}{s}} \leq \\
&\leq \left( \sum_{k=-n+1}^n \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| r\left(t + \frac{\pi}{n}\right) - r(t) \right|^s \frac{\pi}{n} \right)^{\frac{1}{s}} \leq \\
&\leq \frac{\pi}{n} \sup_{t \in \mathbb{R}} \left| \frac{r'(t)}{r(t)} \right| \left( \sum_{k=-n+1}^n \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r^s(t) \frac{\pi}{n} \right)^{\frac{1}{s}}, \quad 1 \leq s < \infty. \quad (144)
\end{aligned}$$

It follows from (122), (127) and (144) that for  $n \geq 4\pi s M$

$$\left\| r\left(t + \frac{\pi}{n}\right) - r(t) \right\|_s \leq (2\pi + 2) \|r\|_s \frac{M}{n}. \quad (145)$$

In view of (141), (143) and (145) for  $n \geq 4\pi s M$  we arrive at the estimate

$$\begin{aligned}
&\frac{1}{2} \|\phi\left(t + \frac{\pi}{n}\right) - \phi(t)\|_s \geq \|\phi\|_s - \frac{5\pi + 2}{\sqrt{2}} \|r\|_s \frac{M}{n} \geq \\
&\geq \|r\|_s \left( \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} - \left( \frac{15\pi + 6}{\sqrt{2}} + 4 \right) \frac{M}{n} \right) > \|r\|_s \left( \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} - 14\pi \frac{M}{n} \right), \quad 1 \leq s < \infty.
\end{aligned}$$

Thus, the validity of formula (118) is established for  $1 \leq s < \infty$ .

Let us prove the relation (46) for  $s = \infty$ . Consider a function  $\phi^*(t)$  such that

$$\phi^*(t) = \phi_k^*(t), \quad \frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}, \quad k = \overline{-n+1, n},$$

where

$$\phi_k^*(t) = g(t_k^*) \cos(nt + \gamma) + h(t_k^*) \sin(nt + \gamma), \quad (146)$$

and points  $t_k^*$ ,  $t_k^* \in [\frac{(k-1)\pi}{n}, \frac{k\pi}{n}]$  are chosen from the condition

$$r(t_k^*) = \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r(t).$$

For the function  $\phi^*(t)$  the following equality takes place

$$\|\phi^*\|_\infty = \|r\|_C. \quad (147)$$

Indeed,

$$\begin{aligned} \|\phi^*\|_\infty &= \max_{-n+1 \leq k \leq n} \operatorname{ess\,sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} |\phi^*(t)| = \\ &= \max_{-n+1 \leq k \leq n} r(t_k^*) \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| \frac{g(t_k^*)}{r(t_k^*)} \cos(nt + \gamma) + \frac{h(t_k^*)}{r(t_k^*)} \sin(nt + \gamma) \right| = \\ &= \max_{-n+1 \leq k \leq n} r(t_k^*) \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| \cos\left(nt + \gamma - \arg(g(t_k^*) + ih(t_k^*))\right) \right| = \\ &= \max_{-n+1 \leq k \leq n} r(t_k^*) \|\cos t\|_C = \|r\|_C. \end{aligned}$$

It is obvious that in view of (147) we obtain

$$\|\phi\|_\infty = \|\phi^*\|_\infty + \Theta_n^{(6)} \|\phi - \phi^*\|_\infty = \|r\|_C + \Theta_n^{(6)} \|\phi - \phi^*\|_\infty, \quad |\Theta_n^{(6)}| \leq 1. \quad (148)$$

Let us find upper estimate for the quantity  $\|\phi - \phi^*\|_\infty$ . By virtue of (45) and (146), for any  $t \in [\frac{(k-1)\pi}{n}, \frac{k\pi}{n}]$  the following equality takes place

$$\begin{aligned} |\phi(t) - \phi_k^*(t)| &= \left| (r(t) - r(t_k^*)) \left( \frac{g(t_k^*)}{r(t_k^*)} \cos(nt + \gamma) + \frac{h(t_k^*)}{r(t_k^*)} \sin(nt + \gamma) \right) + \right. \\ &\quad \left. + r(t) \left( \left( \frac{g(t)}{r(t)} - \frac{g(t_k^*)}{r(t_k^*)} \right) \cos(nt + \gamma) + \left( \frac{h(t)}{r(t)} - \frac{h(t_k^*)}{r(t_k^*)} \right) \sin(nt + \gamma) \right) \right|. \end{aligned} \quad (149)$$

By using (132), the Lagrange theorem and inequality (127), we get

$$\begin{aligned} &\operatorname{ess\,sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| (r(t) - r(t_k^*)) \left( \frac{g(t_k^*)}{r(t_k^*)} \cos(nt + \gamma) + \frac{h(t_k^*)}{r(t_k^*)} \sin(nt + \gamma) \right) \right| \leq \\ &\leq \sqrt{2} \operatorname{ess\,sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} |r(t) - r(t_k^*)| \leq \frac{\sqrt{2}\pi}{n} \sup_{t \in \mathbb{R}} \left| \frac{r'(t)}{r(t)} \right| \|r\|_C \leq \frac{2\sqrt{2}\pi M}{n} \|r\|_C. \end{aligned} \quad (150)$$

Further, it follows from (132), (136) and (137) that

$$\operatorname{ess\,sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r(t) \left( \left| \frac{g(t)}{r(t)} - \frac{g(t_k^*)}{r(t_k^*)} \right| |\cos(nt + \gamma)| + \left| \frac{h(t)}{r(t)} - \frac{h(t_k^*)}{r(t_k^*)} \right| |\sin(nt + \gamma)| \right) \leq$$

$$\leq 3\sqrt{2}\pi \frac{M}{n} \|r\|_C. \quad (151)$$

In view of (149)–(151), we arrive at the estimate

$$\|\phi - \phi^*\|_\infty = \max_{-n+1 \leq k \leq n} \operatorname{ess\,sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} |\phi(t) - \phi_k^*(t)| \leq 5\sqrt{2}\pi \frac{M}{n} \|r\|_C, \quad n \in \mathbb{N}. \quad (152)$$

It follows from (148), (147) and (152) that

$$\|\phi\|_\infty = \|r\|_C \left(1 + \delta_{\infty,n}^{(1)} \frac{M}{n}\right), \quad |\delta_{\infty,n}^{(1)}| \leq 5\sqrt{2}\pi. \quad (153)$$

Let us prove inequality (118) for  $s = \infty$ . By using the inequality (143) for  $s = \infty$ , by applying Lagrange theorem, formulas (127) and (153), we obtain

$$\begin{aligned} & \frac{1}{2} \|\phi(t + \frac{\pi}{n}) - \phi(t)\|_\infty \geq \\ & \geq \|\phi\|_\infty - \frac{1}{\sqrt{2}} \left( \left\| r\left(t + \frac{\pi}{n}\right) - r(t) \right\|_\infty + 3\pi \|r\|_C \frac{M}{n} \right) \geq \\ & \geq \|\phi\|_\infty - \frac{1}{\sqrt{2}} \left( \frac{\pi}{n} \sup_{t \in \mathbb{R}} \left| \frac{r'(t)}{r(t)} \right| \|r\|_C + 3\pi \|r\|_C \frac{M}{n} \right) > \|r\|_C \left(1 - \frac{15\pi}{\sqrt{2}} \frac{M}{n}\right). \end{aligned}$$

Lemma 1 is proved.

**Remark 1.** In proving of lemma 1 we established more exact, than (49) estimates of quantities  $\delta_{s,n}^{(i)}, i = \overline{1,3}$ . Namely, we showed that for  $n \geq \begin{cases} 4\pi s M, & 1 \leq s < \infty, \\ 1, & s = \infty, \end{cases}$  the following estimates hold

$$\begin{aligned} |\delta_{s,n}^{(1)}| & \leq \begin{cases} \sqrt{2}(5\pi + 2) + 4, & 1 \leq s < \infty, \\ 5\sqrt{2}\pi, & s = \infty, \end{cases} \\ -\frac{15\pi + 6}{\sqrt{2}} - 4 & \leq \delta_{s,n}^{(i)} \leq \sqrt{2}(5\pi + 2) + 4, \quad i = 2, 3, \quad 1 \leq s < \infty, \\ -\frac{15\pi}{\sqrt{2}} & \leq \delta_{s,n}^{(i)} \leq 5\sqrt{2}\pi, \quad i = 2, 3, \quad s = \infty. \end{aligned}$$

## References

- [1] STEPANETS, A.I.: Methods of Approximation Theory. VSP: Leiden, Boston (2005).
- [2] STEPANETS', A.I., SERDYUK, A.S., SHIDLICH, A.L.: On some new criteria for infinite differentiability of periodic functions. (Ukrainian, English) Ukr. Mat. Zh. **59**(10), 1399-1409 (2007); translation in Ukr. Math. J. **59**(10), 1569-1580 (2007).
- [3] STEPANETS, A.I., SERDYUK, A.S., SHIDLICH, A.L.: On relationship between classes of  $(\psi, \bar{\beta})$ -differentiable functions and Gevrey classes. (Russian, English) Ukr. Mat. Zh. **61**(1), 140-144 (2009); translation in Ukr. Math. J. **61**(1), 171-177 (2009).
- [4] STEPANETS, A.I., KUSHPEL', A.K.: Convergence rate of Fourier series and best approximations in the space  $L_p$ . (English. Russian original) Ukr. Math. J. **39**(4), 389-398 (1987); translation from Ukr. Mat. Zh. **39**(4), 483-492 (1987).
- [5] KUSHPEL', A.K.: Estimates of the widths of classes of analytic functions. (English. Russian original) Ukr. Math. J. **41**(4), 493-496 (1989); translation from Ukr. Mat. Zh. **41**(4), 567-570 (1989).
- [6] TELYAKOVSKII, S.A.: Approximation of functions of high smoothness by Fourier sums. (English. Russian original) Ukr. Math. J. **41**(4), 444-451 (1989); translation from Ukr. Mat. Zh. **41**(4), 510-518 (1989).
- [7] TEMLYAKOV, V.N.: To the question on estimates of the diameters of classes of infinite- differentiable functions. (Russian) Mat. Zametki **47**(5), 155-157 (1990).
- [8] TEMLYAKOV, V.N.: On estimates of the diameters of classes of infinite- differentiable functions. (Russian) Dokl. razsh. zased. semin. Inst. Prykl. Mat. im. I.N.Vekua **5**(2), 111-114 (1990).
- [9] SERDYUK, A.S.: On one linear method of approximation of periodic functions. (Ukrainian) Zb. Pr. Inst. Mat. NAN Ukr. **1**(1), 294-336 (2004).
- [10] SERDYUK, A.S.: Approximation of classes of analytic functions by Fourier sums in uniform metric. (Ukrainian, English) Ukr. Mat. Zh. **57**(8), 1079-1096 (2005); translation in Ukr. Math. J. **57**(8), 1275-1296 (2005).
- [11] SERDYUK, A.S., STEPANYUK, T.A.: Order estimates for the best approximation and approximation by Fourier sums of classes of infinitely differentiable functions. (Ukrainian. English summary) Zb. Pr. Inst. Mat. NAN Ukr. **10**(1), 255-282 (2013).
- [12] SERDYUK, A.S., STEPANYUK, T.A.: Estimates for the best approximations of the classes of infinitely differentiable functions in uniform and integral metrics. (Ukrainian, English) Ukr. Mat. Zh., **66**(9), 1244-1256 (2014); English translation in Ukr. Math. J., **66**(9), 1393-1407 (2015).
- [13] Kolmogoroff, A.: Zur Grössenordnung des Restgliedes Fourierschen Reihen differenzierbarer Funktionen. (German) Ann. Math.(2), **36**(2), 521-526 (1935).

- [14] NIKOL'SKII, S.M.: Approximation of functions in the mean by trigonometrical polynomials. (Russian. English summary) *Izv. Akad. Nauk SSSR, Ser. Mat.* **10**, 207–256 (1946).
- [15] TELYAKOVSKII, S.A.: Approximation of differentiable functions by partial sums of their Fourier series. (Russian, English) *Mat. Zametki*, **4**(3), 291–300 (1968); English translation: *Mathematical Notes*, **4**(3), 668–673 (1968).
- [16] HRABOVA, U.Z., SERDYUK, A.S.: Order estimates for the best approximations and approximations by Fourier sums of the classes of  $(\psi, \beta)$ -differential functions. (Ukrainian, English) *Ukr. Mat. Zh.*, **65**(9), 1186–1197 (2013); English translation: *Ukr. Math. J.*, **65**(9), 1319–1331 (2014).
- [17] SERDYUK, A.S., STEPANYUK, T.A.: Order estimates for the best approximations and approximations by Fourier sums of the classes of convolutions of periodic functions of low smoothness in the integral metric. *Ukr. Mat. Zh.*, **66**(12), 1658–1675 (2014); English translation: *Ukr. Math. J.*, **65**(12), 1862–1882 (2015).
- [18] STECHKIN, S.B.: An estimate of the remainder term of Fourier series for differentiable functions. (Russian) *Tr. Mat. Inst. Steklova* **145**, 126–151 (1980).
- [19] SERDYUK, A.S., SOKOLENKO, I.V.: Uniform approximation of the classes of  $(\psi, \overline{\beta})$ -differentiable functions by linear methods. (Ukrainian. English summary) *Zb. Pr. Inst. Mat. NAN Ukr.* **8**(1), 181–189 (2011).
- [20] STEPANETS, A.I.: Classification and approximation of periodic functions. Rev., updated and transl. by P. V. Malyshev and D. V. Malyshev. (English) *Mathematics and its Applications* (Dordrecht). 333. Dordrecht: Kluwer Academic Publishers (1995).
- [21] STEPANETS, A.I.: Deviation of Fourier sums on classes of infinitely differentiable functions. (English) *Ukr. Mat. J.*, **36**(6), 567–573 (1984).
- [22] KORNEICHUK N.P.: Exact Constants in Approximation Theory. *Encyclopedia of Mathematics and Its Applications*, Vol. 38, Cambridge Univ. Press, Cambridge, New York (1990).
- [23] BARI, N.K.: A treatise on trigonometric series. Vol. I, II. Authorized translation by M. F. Mullins. (English) Oxford-London-New York-Paris-Frankfurt: Pergamon Press. XXIII, 553 p.; XIX, 508 p. (1964).
- [24] TITCHMARSH, E.C.: Introduction to the Theory of Fourier Integrals (2nd.ed.) Oxford University Press (1948).
- [25] STEPANETS, A.I., RUKASOV, V.I., CHAICHENKO, S.O.: Approximations by the de la Vallee-Poussin sums. (Russian) *Pratsi Instytutu Matematyky Natsional'noi Akademii Nauk Ukrainy. Matematyka ta ii Zastosuvannya* **68**. Kyiv: Instytut Matematyky NAN Ukrainy (2007).

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